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DYNAMIC RESPONSE OF A COMPOSITE BEAM

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Abstract: *Bridges and railroads made of composite laminates are affected by moving loads. Therefore, it is very important to analyze this effect which would find practical applications in engineering designs. This paper explains the theoretical formulation that governs the dynamic response of a composite beam subjected to a moving load. The Mori-Tanaka method for determination of effective material characteristics is used. The governing equations for the laminated composites are explained here.*

Key words: *fibre reinforced composite, laminate, bridge deck, vibrations.*

1. Introduction

The rapid growth in the use of composite materials in structures has required the development of the theory of mechanics of composite materials and the analysis of structural elements made of composite material. Composite materials have higher strength-to-weight and stiffness-to-weight ratios than metals and find many applications such as composite bridge decks. Therefore, it is very important to understand the response of composite bridges to vehicle-induced vibrations.

2. Modelling and analysis of composite laminated beam

The analysis of structural elements can be performed by analytical and semianalytical approaches or by numerical methods. The advantage of analytical solutions is their generality allowing the designer to take into account various design parameters. Analytical solutions may be either closed form solutions or infinite series and may be exact solutions of the governing equations or variational approaches.

However, analytical solutions are restricted to the analysis of simple structural elements. Otherwise numerical methods have to be applied more general for structural analysis [7].

We consider composite laminated beam under lateral loading. The elementary or classical beam theory assumes that the transverse shear strains are negligible and plane cross-sections before bending remain plane and normal to the axis of the beam after bending (Bernoulli-Euler beam theory).

The assumption of neglecting shear strains is valid if the thickness h is small relative to the length l (h/l <1/20). For thick beams (h/l >1/20) the shear deformation theory is used. The governing equations of the shear deformation theory for composite beams are considered. The differential equations will be developed in detail for bending only, the equations for vibration will be described.

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Laminate beams with simple or double symmetric cross-sections are most important in engineering applications. The derivations are therefore limited to straight beams with simple symmetric constant cross-sections which are predominantly rectangular. The bending moments act in a plane of symmetry. Also cross-sections consisting of partition walls in and orthogonal to the plane of bending.

The analysis and results of the classical laminate theory are sufficiently accurate for thin beams. Such beams are often used in civil engineering. For moderately thick beams we have to take into account the shear deformation effects, at least approximately. The theory of laminate beams corresponds then with the Timoshenko's beam theory [1-3].

However, since Timoshenko's beam theory assumed constant shear strains through the thickness h a shear correction factor is required to correct the shear strain energy. In this section we study the influence of transverse shear deformation upon the bending of laminated beams. When it is applied to beams, the first order shear deformation theory is known as Timoshenko's beam theory.

Based upon the kinematical assumption of the first order shear deformation theory the displacements of the beam have the form

$$u(x,z)=\bar{u}(x)-z\varphi(x) \quad (1)$$

$$w(x,z)=\bar{w}(x) \quad (2)$$

with strains

$$\varepsilon_x(x,z)=\bar{\varepsilon}_x(x)+z\kappa_x(x) \quad (3)$$

$$\gamma_{xz}(x,z)=\varphi(x)+w'(x) \quad (4)$$

where

$$\bar{\varepsilon}_x(x)=\frac{\mathrm{d}\bar{u}}{\mathrm{d}x} \quad (5)$$

$$\kappa_x(x)=\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x} \quad (6)$$

When the transverse shear strain is neglected it follows with $\gamma_{xz}=0$ that the relationship is $\varphi(x)=-w'(x)$ and that is the Bernoulli's kinematics.

Consider a laminate beam element consists with N layers (Fig. 1). The layers are symmetrical sequence to the midplane.

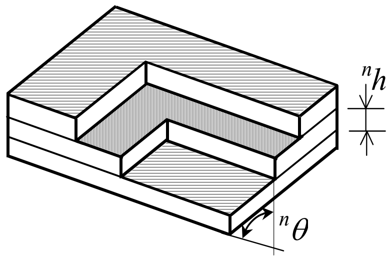

Fig. 1. *Laminate N-layered element*

Constitutive equations are following form

$$M_x=D_{11}\kappa_x \quad (7)$$

$$V_{xz}=k^sA_{55}\gamma_{xz} \quad (8)$$

Substituting the constitutive equations for M_x, V_{xz} into the equilibrium equations of the moments and transverse force resultants results in the following set of governing differential equations for a laminated composite beam subjected to a lateral load p_3 and including transverse shear deformation

$$D_{11}\frac{\partial^2\varphi}{\partial x^2}-k^sA_{55}\left(\varphi+\frac{\partial w}{\partial x}\right)=0 \quad (9)$$

$$k^sA_{55}\left(\frac{\partial\varphi}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+p_3=0 \quad (10)$$

where k^s is transverse shear deformation parameter.

$$D_{11}=\sum_{n=1}^{N}{}^nE_{11}\frac{{}^nz^3-{}^{n-1}z^3}{3} \quad (11)$$

$$A_{55}=\sum_{n=1}^{N}{}^n\mathrm{E}^t_{55}\,{}^nh \quad (12)$$

D_{11} is bending stiffness coefficient,

A_{55} is transverse shear stiffness coefficient,
E_{11} is coefficient of elastic matrix,
More often products and structures are subjected to vehicular dynamic loads. In the linear-elastic range, dynamic effects can be divided into two categories: free vibrations and forced vibrations, and the latter can be further subdivided into one-time events or receiving loads. Mathematically, natural vibration problems are called eigenvalue problems. They are represented by homogeneous equations, for which nontrivial solutions only occur at certain characteristic values of a parameter, from which the natural frequencies are determined. In a natural vibration the displacement field comprises a normal mode.
The shear deformation theory can be used for modeling and analysis of forced vibrations of laminate beams. In the general case of forced vibrations the displacements w, the rotation φ and the transverse load p_3 are functions of x and t. When in-plane loading is not considered but in-plane displacements, rotary and coupling inertia terms have to take into account for unsymmetrical laminate beam.
The governing equations for the calculation of natural frequencies of especially orthotropic beams made of symmetric layers without coupling effect

$$D_{11}\frac{\partial^2\varphi}{\partial x^2}-k^s A_{55}\left(\varphi+\frac{\partial w}{\partial x}\right)-I_2\frac{\partial^2\varphi}{\partial t^2}=0 \tag{13}$$

$$k^s A_{55}\left(\frac{\partial\varphi}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)-\rho_m h\frac{\partial^2 w}{\partial t^2}=0 \tag{14}$$

$$\rho_m=\frac{1}{h}\sum_{n=1}^{N}{}^{n}\rho({}^{n}z-{}^{n-1}z),$$
$$I_2=\frac{\rho_m h^3}{12}=\frac{1}{3}\sum_{n=1}^{N}{}^{n}\rho({}^{n}z-{}^{n-1}z)^3 \tag{15}$$

where

ρ_m is mass density of the laminate

I_2 is rotational inertia term.

3. Transient dynamic analysis

This type of analysis is also known as time-history analysis. This method is generally used to determine response due to time varying loads. Through this analysis, we can find the time varying stresses, strains, and deflections produced when a system responds to the transient loads. A transient analysis is more complex and time consuming method compared to a static analysis, as it requires more engineering input data and better understanding of the system response. An analyst must have a good insight of the problem involved in the analysis.
Solving of force vibration means to solve the equations of motion

$$\boldsymbol{m}_D.\ddot{\boldsymbol{v}}(t)+\boldsymbol{b}\dot{\boldsymbol{v}}(t)+\boldsymbol{k}\boldsymbol{v}(t)=\boldsymbol{F}(t) \tag{16}$$

The equations we can solve by numerical methods. The program MATLAB serves procedure for the solving the differential equations of the first order by the Runge-Kutta-Fehlberg method. Therefore the equations of motion are the second order we can transform them by the applicable substitution to the first order equations. From the equation (16) we get

$$\ddot{\boldsymbol{v}}(t)=\left(\boldsymbol{F}(t)-\boldsymbol{b}\dot{\boldsymbol{v}}(t)-\boldsymbol{k}\boldsymbol{v}(t)\right)/\,\boldsymbol{m}_D \tag{17}$$

After substitution we have

$$\boldsymbol{v}(t)=\boldsymbol{y}_1(t)$$
$$\dot{\boldsymbol{v}}(t)=\boldsymbol{y}_2(t)$$
$$\ddot{\boldsymbol{v}}(t)=\dot{\boldsymbol{y}}_2(t) \tag{18}$$

Than we solve the system of equations the first order

$$\dot{\boldsymbol{y}}_1(t)=\boldsymbol{y}_2(t)$$
$$\dot{\boldsymbol{y}}_2(t)=\ddot{\boldsymbol{v}}(t)=f(t,v,\dot{v}) \tag{19}$$

Modeling of a beam made of composite materials is a more difficult task. Special attention has to be paid in defining the material properties, orientations of the layers and the element coordinate systems. Boundary conditions are the constraints and loads that can simulate the effect of the environment surrounding a body. Loads are applied in the form of forces and temperatures. Since, improper application of boundary conditions can create problems such as increased stiffness, rigid body motion, and high local stresses. In the present paper, the simply supported beam is used for analysis. A symmetric cross-ply laminated beam, $[0/90]_{25s}$, made of boron/epoxy is analyzed as a cross-section of the bridge. In this type of analysis a discrete beam model is presented. In the discrete beam model, the bridge is modeled as one lumped mass connected by massless beam elements. In this beam model, the effects of shear deformation and rotary inertia are neglected. The beam has a constant cross section and mass per unit length with damping. The vehicle is assumed to move from one end to the other with constant velocity. The model of vehicle consists of two masses with damping (Fig. 2).

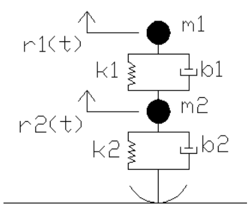

Fig. 2. *The model of vehicle made of two masses with damping*

The equations of vehicle motion have the form

$$m_1\ddot{r}_1(t)+b_1(\dot{r}_1(t)-\dot{r}_2(t))+k_1(r_1(t)-r_2(t))=0 \quad (19)$$

$$m_2\ddot{r}_2(t)-b_1(\dot{r}_1(t)-\dot{r}_2(t))+b_2(\dot{r}_2(t)-\dot{v}_x(t))-$$
$$-k_1(r_1(t)-r_2(t))+k_2(r_2(t)-v_x(t))=0 \quad (20)$$

The equation of bridge motion has the form

$$m\ddot{y}(t)+2m\omega_b\dot{y}(t)+ky(t)=F(t) \quad (21)$$

The equations of the road profile are assumed

$$v_x(t)=y_x(t)+h(t)=\phi_x(t)y(t)+h(t) \quad (22)$$

$$\dot{v}_x(t)=\dot{y}_x(t)+\dot{h}(t)=\phi_x(t)\dot{y}(t)+\dot{h}(t) \quad (23)$$

The shape function of the deflection curve is

$$\phi_x(t)=\sin\left(\frac{\pi c t}{l}\right)=\sin\omega t \quad (24)$$

where c is speed of the vehicle in [m/s]. There will be avoided the next substitution

$$r_1(t)=ys_1(t) \qquad \dot{r}_1(t)=ys_2(t)$$
$$r_2(t)=ys_3(t) \qquad \dot{r}_2(t)=ys_4(t)$$
$$y(t)=ys_5(t) \qquad \dot{y}(t)=ys_6(t) \quad (25)$$

Then we can solve six differential equations of the first order

$$\dot{y}s_1(t)=ys_2(t) \qquad \dot{y}s_2(t)=\ddot{r}_1(t)$$
$$\dot{y}s_3(t)=ys_4(t) \qquad \dot{y}s_4(t)=\ddot{r}_2(t)$$
$$\dot{y}s_5(t)=ys_6(t) \qquad \dot{y}s_6(t)=\ddot{y}(t) \quad (26)$$

4. Solution, Discussion and Results

A symmetric cross-ply laminated beam, $[(0/90)_{25}]_s$, made of boron/epoxy is analyzed next. The thickness of the laminate bridge deck is 120 mm. The geometric and material properties of this model are listed in Table 1. The simply supported beam is used for analysis (Fig. 3).

Parameters of the vehicle T148:
m_1= 18000 kg, k_1= 3145762 N/m,
b_1= 260197 kg/s,
m_2= 2120 kg, k_2= 9600000 N/m,
b_2= 10987.2 kg/s,
g= 9.81 m/s^2,
V= 20, 40, 60, 80, 100, 120 km/h.
Parameters of the bridge:
m= 6442 kg/m, I= 0.231099 m^4,
E= 1.15.10^{11} Pa, L= 37 m,
ω_b= 0.23321 rad/s.
Boundary conditions:

t=0, $r_1(0)$= 0.02 m, $r_2(0)$= 0.0033 m,
$y(0)$=0.0 m, $\dot{r}_1(0)$= 0.0 m/s.

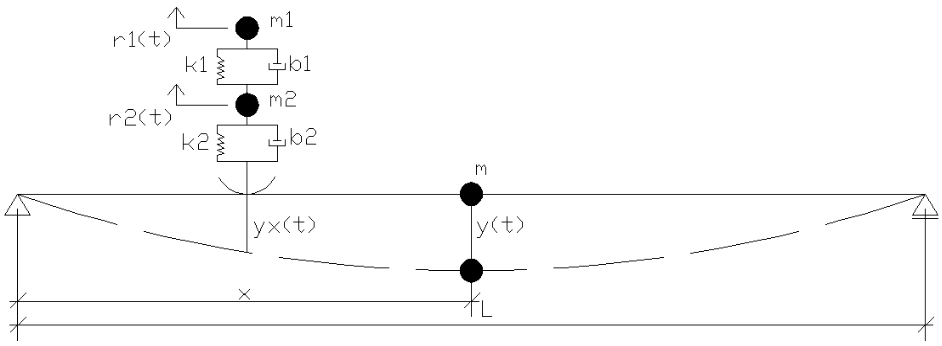

Fig. 3. *Model of bridge* [4]

Material properties of composite laminate Table 1

Property	Value
Mass density of the composite, ρ	2100 kg/m^3
Longitudinal modulus, E_1	214 GPa
Transverse modulus, E_2	18.7 GPa
Longitudinal shear modulus, G_{12}	4 GPa
Major in-plane Poisson's ratio, ν_{12}	0.27
Fiber volume fraction, ξ	0.55
Effective moduli of laminate $[(0/90)_{25}]_s$, $E_x = E_y$	115 GPa
Effective shear modulus of laminate $[(0/90)_{25}]_s$, G_{xy}	4.8 GPa
Effective in-plane Poisson's ratio of laminate, ν_{xy}	0.035
The natural frequencies of the vehicle	1.819 Hz, 12.386 Hz
The first natural frequency of the bridge	2.334 Hz

Dynamic magnification factors for moving load on composite beam Table 2

Velocity (km/h)	$t_{max.}$ (s)	t (s)	Dynamic deflection (m)	Dynamic magnification factor
20	3.3419	6.66	0.00782	1.00057
40	1.6985	3.33	0.00796	1.01846
60	0.8144	2.22	0.00816	1.04167
80	**0.7973**	**1.665**	**0.00907**	**1.16115**
100	0.7731	1.33	0.009	1.15153
120	0.3671	1.11	0.00895	1.14478

5. Conclusion

The use of composite materials in the modern engineering applications has been increasing rapidly. Bridges, aerospace structures are few examples of their application. Steel bridges are replaced by composite materials due to their superior qualities like higher strength-to-weight ratio. Bridge structures are constantly being exposed to various types of loads. The major loads that influence the life of a bridge is dynamic moving loads.

Effective material characteristics were established using the program HELP [8-10]. This program works under Mori-Tanaka method.

The modal analysis and forced vibration analysis of laminated composite beams under the effect of moving loads using the program MATLAB [4-6] was investigated. The program MATLAB serves procedures for the solving the differential equations of the first order by the Runge-Kutta-Fehlberg methods.

The dynamic magnification factors of composite beams were calculated (Table 2). The maximum dynamic magnification factor occurs at the velocity of 80km/h.

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