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THE DISCRETIZATION OF THE LIMIT OF A BOUNDARY VALUE DIFFUSION PROBLEM IN A PERFORATED DOMAIN

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Abstract: *This paper presents the approximation of a boundary value problem of diffusion in a perforated domain with very small holes. For the beginning, we homogenize the problem using two-scale convergence and than, the limit problem is discretized applying the finite element method and the operator-splitting method.*
Keywords: *perforated domain with small holes, homogenization, operator splitting, finite element method.*

1. INTRODUCTION

In this paper we resume the idea from the article [3], the different is that the domain is perforated with holes with a diameter much smaller than the period which are distributed in the domain. In the second, third and fourth sections of the article we discuss the homogenization of the non-stationary diffusion problem with Robin conditions. The limit problem obtained is a non-stationary convection-diffusion problem considered on the cylindrical domain $Q = \Omega \times (0,T)$, where Ω is the initial fixed domain without holes.

The novelty of the present article is the approximation of the limit problem on $Q = \Omega \times (0,T)$. The fifth section presents the spatial discretization of the locale problems which were obtained after the homogenization process from the fourth section, using the finite element method. In the sixth section we approximated the limit problem of the section four, combining the operator splitting – the Glowinski's scheme of the fractional step for the temporal discrimination (decomposition of the convection-diffusion operator) with the finite element method for the spatial discretization. The last section presents the result of the convergence.

2. THE PERFORATED DOMAIN

We consider the open and bounded domain $\Omega \subset R^n$, with the Lipschitz border $\partial\Omega$, the reference cell $Y = (o,l_1)\times(o,l_2)\times\cdots\times(o,l_n)$ and let be an open domain $S \subset Y$ so that $\overline{S} \subset Y$ with smooth border ∂S. Let be $r_\varepsilon << \varepsilon$ so that $\lim\limits_{\varepsilon\to 0}\frac{r_\varepsilon}{\varepsilon} = 0$ and $\lim\limits_{\varepsilon\to 0}\frac{\varepsilon^n}{r_\varepsilon^{n-2}} = 0$. We consider $\Im(r_\varepsilon S)$ the translated of $r_\varepsilon \overline{S}$ with the form $\left(\varepsilon kl + r_\varepsilon \overline{S}\right)$, where $k \in Z^n$, $kl = (k_1 l_1, k_2 l_2, ..., k_n l_n)$, these translated representing the micro holes from R^n. We denote by

$S_\varepsilon = \bigcup\limits_{k\in K_\varepsilon}\left(\varepsilon kl + r_\varepsilon \overline{S}\right)$, where $K_\varepsilon = \left\{k \in Z^n \middle| \left(\varepsilon kl + r_\varepsilon \overline{S}\right)\cap\Omega \neq \Phi\right\}$, S_ε represented the finite reunion of the holes from Ω, which can intersect $\partial\Omega$.

We defining now the perforated domain $\Omega_\varepsilon = \Omega \setminus \overline{S}_\varepsilon$ where the holes are distributed with the period ε and the diameter r_ε is much smaller than ε.

3. THE STATE OF THE PROBLEM

We consider the following non-stationary diffusion problem in the perforated domain Ω_ε.

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - div(A_\varepsilon \nabla u_\varepsilon) + \mu_\varepsilon u_\varepsilon = f_\varepsilon \quad \text{in} \quad \Omega_\varepsilon \times (0,T) \\ (A_\varepsilon \nabla u_\varepsilon)\nu_\varepsilon + \alpha_\varepsilon u_\varepsilon = \dfrac{\varepsilon^n}{r_\varepsilon^{n-1}} g_\varepsilon \quad \text{on} \quad \Sigma_\varepsilon \times (0,T) \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ u_\varepsilon(0) = u_\varepsilon^0 \quad \text{on} \quad \Omega_\varepsilon \end{cases} \tag{1}$$

under the following conditions:

i) $f_\varepsilon \in L^2(\Omega_\varepsilon \times (0,T))$, $g_\varepsilon \in L^2(\Sigma_\varepsilon \times (0,T))$, where $\Sigma_\varepsilon = \partial S_\varepsilon$ represents the border of the holes from the domain Ω_ε, $\partial S = \Sigma$. The estimation $\|f_\varepsilon\|_{L^2(\Omega_\varepsilon)} + \sqrt{\varepsilon}\|g_\varepsilon\|_{L^2(\Sigma_\varepsilon)} \le c$ is true, where c is a positive constant, independent of ε.

ii) $A \in \left(L_{per}^\infty(Y)\right)^{n\times n}$, $m|\xi|^2 \le A_{ij}(y)\xi_i\xi_j \le \beta|\xi|^2$, $\forall \xi \in R^n$ a.e. $y \in Y$.

iii) $\mu \in L_{per}^\infty(Y)$, $\int_Y \mu(y)dy \ge \mu_0 > 0$; $\alpha \in L_{per}^2(\Sigma)$ so that $\int_\Sigma \alpha(y)d\sigma(y) = 0$, $u_\varepsilon^0 \in L^2(\Omega_\varepsilon)$.

We denote by $A_\varepsilon(x) = A\left(\dfrac{x}{\varepsilon}\right)$, $\mu_\varepsilon(x) = \mu\left(\dfrac{x}{\varepsilon}\right)$, $\alpha_\varepsilon(x) = \alpha\left(\dfrac{x}{r_\varepsilon}\right)$, $g_\varepsilon(x) = g\left(x, \dfrac{x}{r_\varepsilon}\right)$ with $g \in L^2(\Omega \times \Sigma)$ and ν_ε is the external normal to Σ_ε.

4. THE HOMOGENIZATION

Using the homogenization method of the multiple scales like in the paper [3] and proving the convergence of the homogenization process with two-scale convergence method like in [1] – where we taking into account by the convergences

$$\frac{\varepsilon^n}{r_\varepsilon^{n-1}} \chi_{\Omega_\varepsilon} u_\varepsilon \xrightarrow{2s} u(x,t), \quad \frac{\varepsilon^n}{r_\varepsilon^{n-1}} \chi_{\Omega_\varepsilon} \nabla u_\varepsilon \xrightarrow{2s} \nabla_x u(x,t) + \nabla_y U(x,y,t),$$

respectively [7]

$$\lim_{\varepsilon\to 0} \int_{\Sigma_\varepsilon} \frac{\varepsilon^n}{r_\varepsilon^{n-1}} g_\varepsilon \varphi d\sigma^\varepsilon(x) = \frac{1}{|Y|}\int_\Omega \left[\int_\Sigma g(x,y,t)d\sigma(y)\right]\varphi(x)dx, \ \forall \varphi \in H_0^1(\Omega)$$

we obtain the homogenized problem with convection

$$\begin{cases} \dfrac{\partial u}{\partial t} - div\left(A^{eff}\nabla u(x,t)\right) + B\nabla u(x,t) + \lambda u(x,t) = F(x,t) \quad \text{in} \quad \Omega \times (0,T) \\ u = 0 \quad \text{on} \quad \partial\Omega \times (0,T) \\ u(0) = u_0 \quad \text{on} \quad \Omega \end{cases} \tag{2}$$

where $\chi_{\Omega_\varepsilon} u_\varepsilon^0 \xrightarrow{2s} u_0$ and the other constants are

$$a_{ik}^{eff} = \frac{1}{|Y|}\int_Y a_{ij}(y)\frac{\partial(\chi_k(y)+y_k)}{\partial y_j}dy \tag{3}$$

where the correctors χ_k satisfies the local problem

$$\begin{cases} -div_y\left[A(y)\nabla_y\left(y_j-\chi_j(y)\right)\right]=0 \quad \text{in} \quad Y \\ \chi_j \ \text{ is Y - periodically}, \quad j=\overline{1,n}. \end{cases} \tag{4}$$

$$b_i = -\frac{1}{|Y|}\int_{Y^*} a_{ij}(y)\frac{\partial \gamma}{\partial y_j}(y)dy + \frac{1}{|Y|}\int_\Sigma \alpha(y)\chi_i(y)d\sigma(y), \quad i=\overline{1,n}, \tag{5}$$

where $B=(b_i)_{1\le i\le n}$ is the convection vector.

$$\lambda = \overline{\mu} + \int_\Sigma \alpha(y)\gamma(y)d\sigma(y), \tag{6}$$

where $\overline{\mu} = \frac{1}{|Y|}\int_Y \mu(y)dy$, and γ satisfies the local problem:

$$\begin{cases} -div_y\left[A(y)\nabla\gamma(y)\right]=0 \quad \text{in} \quad Y^*, \\ \left[A(y)\nabla\gamma(y)\right]\nu = -\alpha(y) \quad \text{on} \quad \Sigma, \\ \gamma \ \text{ is Y - periodically}, \end{cases} \tag{7}$$

$$F(x,t) = \frac{1}{|Y|}\left[\int_Y f(x,y,t)dy + \int_\Sigma g(x,y,t)d\sigma(y)\right]. \tag{8}$$

5. THE SPATIAL DISCRETIZATION

The spatial discretization of the local problems is made with the finite element method. Because the local problems (4) and (7) are considered on two different domains Y, respectively Y^*, and these two problems are independent of each other, we will consider two different triangulations: $\Im_{h/2}$ on Y and respectively $\Im_{h/4}$ on Y^*, with $h>0$. In the following figures we present these two triangulations.

About the discretizated coefficients $a_{ik,h}^{eff}$, $b_{i,h}$ and λ_h we apply a quadrature scheme.

We consider the bidimensional case and we choose $Y=[0,1]^2$, $S=\left(\frac{3}{8},\frac{5}{8}\right)^2$.

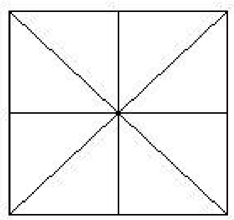

Figure 1: $\Im_{h/2}$

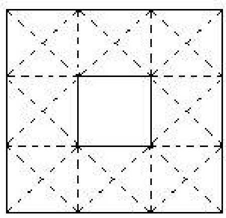

Figure 2: $\Im_{h/4}$

We consider the finite dimensional spaces

$$V_{h/2} = \left\{ v_h \middle| v_h \in C^0\left(\overline{Y}\right), v_{h|K} \in P_{11}, \forall K \in \Im_{h/2} \right\},$$

$$P_{11} = \left\{ p(x_1, x_2) = \sum_{0 \le i, j \le 1} c_{ij} x_1^i x_2^j, c_{ij} \in R \right\},$$

$$W_{h/4} = \left\{ w_h \middle| w_h \in C^0\left(Y^*\right), w_{h|K} \in P_{11}, \forall K \in \Im_{h/4} \right\},$$

and the subspations

$$V_{per,h/2} = \left\{ v_h \in V_h \middle| v_h(0, y_2) = v_h(1, y_2), v_h(y_1, 0) = v_h(y_1, 1), \forall y_1, y_2 \in [0,1] \right\},$$

$$W_{per,h/4} = \left\{ w_h \in W_h \middle| w_h \text{ is periodically} \right\}.$$

In this case, the spatial discretisation of the local problems (4) and (7) is

To find $\chi_{j,h} \in V_{per,h/2}$ so that:

$$\int_Y A_{h/2}(y) \nabla_y \chi_{j,h}(y) \cdot \nabla_y v_h(y) dy = \int_Y A_h(y) \cdot e_j \cdot \nabla_y v_h(y) dy, \forall v_h \in V_{per,h/2} \tag{9}$$

respectively,

To find $\gamma_h \in W_{per,h/4}$ so that:

$$\int_{Y^*} A_{h/4}(y) \nabla_y \gamma_h(y) \cdot \nabla_y w_h(y) dy = \int_\Sigma \alpha_h(y) w_h(y) d\sigma(y), \forall w_h \in W_{per,h/4} \tag{10}$$

where $A_{h/2}$, $A_{h/4}$ represents the approximations of the matrix $A(y)$ relative to $\Im_{h/2}$, respectively $\Im_{h/4}$. The relations (9) and (10) represent linear algebraic systems.

The discretizated coefficients are obtained from the equations (3), (5) and (6):

$$a_{ik,h}^{eff} = \sum_{K \in \Im_{h/2}} \int_K a_{ij,h}(y) \frac{\partial\left(\chi_{k,h}(y) + y_k\right)}{\partial y_j} dy,$$

$$b_{i,h} = -\sum_{K \in \Im_{h/4}} \int_K a_{ij,h}(y) \frac{\partial \gamma_h}{\partial y_j} dy + \sum_{\substack{K \in \Im_{h/4} \\ K \cap \Sigma \neq \Phi}} \int_{\partial K \cap \Sigma} \alpha_h(y) \chi_{i,h}(y) d\sigma(y), \tag{11}$$

$$\lambda_h = \sum_{K \in \Im_{h/2}} \int_K \mu_h(y) dy + \sum_{\substack{K \in \Im_{h/4} \\ K \cap \Sigma \neq \Phi}} \int_{\partial K \cap \Sigma} \alpha_h(y) \gamma_h(y) d\sigma(y),$$

where α_h and μ_h are the approximations of the functions α (to $\Im_{h/4}$) and μ (to $\Im_{h/2}$). We apply the quadrature schemes to integrals.

Regarding the free term $F(x,t)$, the two integrals of relation (8) are calculated using the quadrature scheme.

6. THE DISCRETIZATION

The discretization of the problem (2) using the operator splitting method and the finite element method.

We discretize the global problem (2) combining the operator splitting – the Glowinski's scheme of the fractional step with the finite element method. We consider Ω a bounded poligonal domain from R^2. Let be $\Im_h$ a triangulation of Ω. We introduce the spaces:

$W_h = \left\{ v_h \in C^0\left(\overline{\Omega}\right) \middle| v_{h/T} \in P_{11}, \forall T \in \Im_h \right\}$, where P_{11} is the space of polynomials in two variables with degree at most one;

$$W_{0h} = \left\{ v_h \in W_h \middle| v_h = 0 \text{ on } \partial\Omega \right\}.$$

We consider the partition of the interval $[0,T]$:

$$0 = t^0 < t^1 = \Delta t < \cdots < t^n = n\Delta t < t^{n+1} = (n+1)\Delta t < \cdots < t^N = N\Delta t = T,$$

$\Delta t = \dfrac{T}{N}$ and $t^{n+\theta} = (n+\theta)\Delta t$, $\theta \in \left\{0, \dfrac{1}{3}, \dfrac{2}{3}, 1\right\}$.

We have the following scheme: we denote by $u^0 = u_0$ and let $u_h{}^0 = (u_0)_h$ an approximation of u_0. We assume that u_h^n is known and we consider the following discrete variational problems:

Let find $u_h^{n+\frac{1}{3}}, u_h^{n+\frac{2}{3}} \in W_h$ and $u_h^{n+1} \in W_{0h}$ so that

$$3\int_\Omega \frac{u_h^{n+\frac{1}{3}} - u_h^n}{\Delta t} v_h dx + \int_\Omega A_h^{eff} \nabla u_h^{n+\frac{1}{3}} \nabla v_h dx = \int_\Omega \left(F_h^n - B_h \nabla u_h^n - \lambda_h u_h^n\right) v_h dx, \forall v_h \in W_h,$$

$$3\int_\Omega \frac{u_h^{n+\frac{2}{3}} - u_h^{n+\frac{1}{3}}}{\Delta t} v_h dx + \int_\Omega B_h \nabla u_h^{n+\frac{2}{3}} v_h dx + \int_\Omega \lambda_h u_h^{n+\frac{2}{3}} v_h dx =$$

$$= \int_\Omega \left(F_h^{n+\frac{1}{3}} v_h - A_h^{eff} \nabla u_h^{n+\frac{1}{3}} \nabla v_h \right) dx, \forall v_h \in W_h, \tag{12}$$

$$3\int_\Omega \frac{u_h^{n+1} - u_h^{n+\frac{2}{3}}}{\Delta t} v_h dx + \int_\Omega A_h^{eff} \nabla u_h^{n+1} \nabla v_h dx =$$

$$= \int_\Omega F_h^{n+\frac{2}{3}} v_h dx - \int_\Omega B_h \nabla u_h^{n+\frac{2}{3}} v_h dx - \int_\Omega \lambda_h u_h^{n+\frac{2}{3}} v_h dx, \forall v_h \in W_{0h},$$

where $F_h^n = F_h(t^n)$ in Ω, $F_h^{n+\frac{1}{3}} = F_h\left(t^{n+\frac{1}{3}}\right)$, $F_h^{n+\frac{2}{3}} = F_h\left(t^{n+\frac{2}{3}}\right)$, and F_h is an approximation of F relative to T_h and $t^{n+\theta} = (n+\theta)\Delta t$, $\theta \in \left\{0, \dfrac{1}{3}, \dfrac{2}{3}, 1\right\}$.

Therefore, the switching from u_h^n to u_h^{n+1} is made passing through $u_h^{n+\frac{1}{3}}$ and $u_h^{n+\frac{2}{3}}$, practically breaking the interval (t^n, t^{n+1}) with the intermediary points $t^{n+\frac{1}{3}}$ and $t^{n+\frac{2}{3}}$, where $t^{n+\theta} = (n+\theta)\Delta t$, $\theta \in \left\{0, \dfrac{1}{3}, \dfrac{2}{3}, 1\right\}$, $n \in \{0,1,\ldots,N-1\}$, where we have the partition of the interval $[0,T]$:

$$0 = t^0 < t^1 < \cdots < t^n < t^{n+1} < \cdots < t^N = T, \quad t^n = n\Delta t, \quad t^N = N\Delta t = N\frac{T}{N} = T$$

and also braking the convection-diffusion operator, we denote by $\mathsf{L}_1, \mathsf{L}_2$ the operators:

$$\begin{cases} \mathsf{L}_1 = -div\left(A^{eff}\nabla\right) \\ \mathsf{L}_2 = B\nabla + \lambda. \end{cases}$$

This is the decomposition of the next operator

$$\mathsf{L} = -div\left(A^{eff}\nabla\right) + B\nabla + \lambda.$$

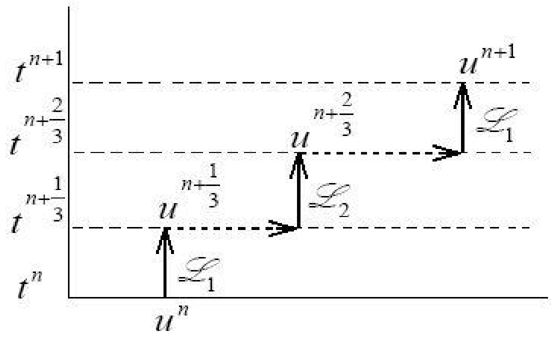

Figure 3:

7. CONCLUSION

In the section 6 we combined the finite element method with the operator splitting for the convection-diffusion operator and for the breaking of the interval (t^n, t^{n+1}), but before we partitioned the time interval $(0,T)$ such that

$$0 = t^0 < t^1 = \Delta t < \cdots < t^n = n\Delta t < t^{n+1} = (n+1)\Delta t < \cdots < t^N = N\Delta t = T,\ \Delta t = \frac{T}{N}.$$

By the other hand, from the ellipticity of the coefficients and the Schwars's inequality we find the estimation $\sum_{n=0}^{N-1} \left\| u_h^{n+\theta} \right\|_h^2 \leq \text{constant}$, for $\theta \in \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}$, and $\|\cdot\|_h$ is the norm on W_{0h}- the space which is the approximation of $H_0^1(\Omega)$.

Also, we obtain the convergence

$$u_h^{n+\theta} \xrightarrow[h \to 0]{} u(\cdot, t^{n+\theta}) \text{ strong in } L^2(\Omega),\ \theta \in \left\{0, \frac{1}{3}, \frac{2}{3}, 1\right\}.$$

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