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# A RELATION FOR CALCULATING THE EIGENVALUES FOR A CONTINOUS THREE-SPAN BEAM WITH CLAMPED-HINGED ENDS

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***Abstract:*** *The paper propose a method to be used for calculating the eigenvalues and the natural frequencies for a three spam beam that is fixed at one end and hinged on the other, when the intermediate supports are located any ware along the length of the beam. It was assumed that each span of the continuous beam follows the Euler-Bernoulli beam theory. The case of a three-span beam has been taken into consideration, as this is a very common structure used in practice. By imposing the correct boundary conditions and by using the frequency and normalized mode shape equations, the eigenvalues and the modal function are obtained.*
***Keywords:*** *natural frequency, eigenvalues, mode shape, three-span beam*

## 1. INTRODUCTION

The current research describes the evolution of the eigenvalues and natural frequencies for a three-span beam clamped at one end and hinged at the other to which two intermediate support hinges have been added and iteratively moved from one beam end to the other. In practice, multiple-span beams are largely used in engineering applications including bridges, cableways, cranes and marine structures, that always have at least one clamped end and one or more supports. The use of intermediate stiffeners helps improve the overall rigidity and damping ratios of structures [1]. Beams that have more than one span are known as continuous beams [2].
The equation of motion for a continuous beam in bending vibration is used to describe the behavior of the three spans [3]. The boundary conditions were defined, in addition to the continuity condition imposed at the intermediate hinges. The eigenvalues can be used to get the structure’s natural frequencies, of interest being the first several values; henceforth six weak-axis bending modes are considered. We assume that the beam material follows Hooke’s law, meaning that the beam is homogeneous and isotropic. Many studies have been made for obtaining the solution for the boundary conditions for transversal vibrating beams formulated in terms of the partial differential equation of motion, for instance the studies done by Traill-Nash and Collar [4, 5], but they derived the frequency equations only for four models. The Rayleigh beam theory [6] offers a marginal improvement for depicting the natural frequencies and eigenvalues by including the rotary inertia effect of the cross-section. There are numerous approaches for modelling the dynamic behavior of continuous beams, including: Timoshenko models, wave-propagation approach, Rayleigh-Ritz procedure, and the finite element method [7]. This paper presents a method to calculate the eigenvalues for continuous beams with high accuracy.

## 2. EIGENVALUES APPROACH

The cross-section of the continuous beam is rectangular and the natural frequencies for the $n^{th}$ vibration mode is given by the equation:

$$f_n = \frac{a_n^2}{2\pi}\sqrt{\frac{E \cdot I}{m \cdot L^4}} \tag{1}$$

where:
$f_n$ [Hz] is the natural frequency;
$a_n$ is the eigenvalues for a specific mode of vibration;
$E$ [N/m$^2$] is the elasticity modulus;
$I$ [m$^4$] is the moment of inertia;
$m$ [kg] is the beam mass;
$L$ [m] is the beam length;
$n$ = $n^{th}$ vibration mode number;

In order to depict the eigenvalues of a beam on multiple supports, the spans between each pair of supports is considered as a separate beam [8, 9]. From eq. 1 the natural frequencies for the continuous beam can be determined by knowing eigenvalues, beam geometry and material properties.

For any support type by calculating the eigenvalues, the natural frequencies, vibration mode shapes functions and mode shapes can be determined.

The current research considers a continuous beam clamped at one end and hinged on the other, supported with two intermediate hinges, that means three spans (fig. 1). It is known that the deflection and the slope is zero for clamped support, deflection and bending moment is zero for the end hinge of the beam. Since the beam is continuous, the slope and bending moment to the left and to the right of the intermediate supports are the same. Also, the deflection is zero for the intermediate supports. The length of the beam is normalized so that it is considered $L$=1.

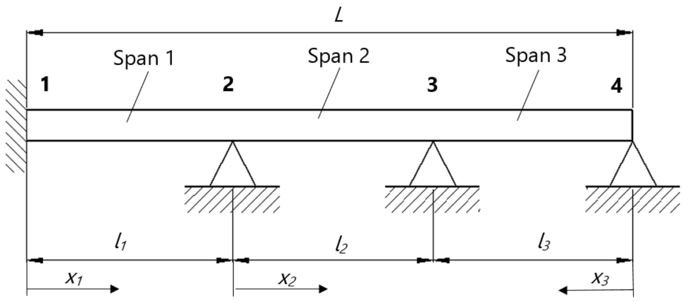


**Figure 1:** Continous beam having three spans

For each support (noted in fig. 1 with 1, 2, 3 and 4), the boundary conditions can be written as:

$$
1.\ \begin{cases} W_1(0)=0 \\ \dfrac{dW_1(0)}{dx}=0 \end{cases}
\qquad
2.\ \begin{cases} W_1(l_1)=0 \\ W_2(0)=0 \\ \dfrac{dW_1(l_1)}{dx}=\dfrac{dW_2(0)}{dx} \\ \dfrac{d^2W_1(l_1)}{dx^2}=\dfrac{d^2W_2(0)}{dx^2} \end{cases}
\qquad
3.\ \begin{cases} W_2(l_2)=0 \\ W_3(l_3)=0 \\ \dfrac{dW_2(l_2)}{dx}=-\dfrac{dW_3(l_3)}{dx} \\ \dfrac{d^2W_2(l_2)}{dx^2}=\dfrac{d^2W_3(l_3)}{dx^2} \end{cases}
\qquad
4.\ \begin{cases} W_3(0)=0 \\ \dfrac{d^2W_3(0)}{dx^2}=0 \end{cases}
\tag{2}
$$

where, the mode shape function or normal mode of span can be expressed generally:

$$W_i(x_i)=A_i\sin(a_nx_i)+B_i\cos(a_nx_i)+C_i\sinh(a_nx_i)+D_i\cosh(a_nx_i) \tag{3}$$

where $i$ is the boundary and continuity conditions of the i$^{th}$ span, i.e. i = 1, 2, 3 represents the number of spans. The integration coefficients: $A_i$, $B_i$, $C_i$, $D_i$ are determined by solving the system of equations (2). For a certain configuration of the continuous beam between two consecutive supports, the notations (4) are introduced which have a constant value:

$$
\begin{aligned}
&(1-2)\ \begin{cases} Z_{11}=\cos(a_nl_1)-\cosh(a_nl_1)+\dfrac{\sin(a_nl_1)-\sinh(a_nl_1)}{\cos(a_nl_1)-\cosh(a_nl_1)}\left[\sin(a_nl_1)+\sinh(a_nl_1)\right] \\ Z_{12}=\sin(a_nl_1)+\sinh(a_nl_1)-\dfrac{\sin(a_nl_1)-\sinh(a_nl_1)}{\cos(a_nl_1)-\cosh(a_nl_1)}\left[\cos(a_nl_1)+\cosh(a_nl_1)\right] \end{cases} \\
&(2-3)\ \begin{cases} Z_{21}=1-\cos(a_nl_2)\cosh(a_nl_2) \\ Z_{22}=\cos(a_nl_2)\sinh(a_nl_2)-\sin(a_nl_2)\cosh(a_nl_2) \\ Z_{23}=2\sin(a_nl_2)\sinh(a_nl_2) \end{cases} \\
&(3-4)\ \begin{cases} Z_{31}=\cos(a_nl_3)-\dfrac{\sin(a_nl_3)}{\sinh(a_nl_3)}\cosh(a_nl_3) \\ Z_{32}=2\sin(a_nl_3) \end{cases}
\end{aligned}
\tag{4}
$$

Finally, the frequency equation (5) is obtained, whose solution represents eigenvalues $a_n$, for a continuous beam with three spans:

$$(Z_{12}\cdot Z_{21}+Z_{11}\cdot Z_{22})\cdot Z_{32}+(Z_{12}\cdot Z_{22}+Z_{11}\cdot Z_{23})\cdot Z_{31}=0 \tag{5}$$

In this form, relation (5) represents the generalized expression of the frequency equation for continuous beams with three openings for any type of support at the ends.
When using the relation (5), the boundary conditions at the ends of the beam must be taken into account, respectively the type of considered support, in other words, the constants $Z_{12}$, $Z_{22}$, $Z_{31}$, $Z_{32}$ will be customized for each type of support.

## 3. MODE SHAPE EQUATION AND INTEGRATION CONSTANTS

By solving the system (2), the integration constants (6) are obtained and the modal functions (7) for each span, as continuous functions for the whole structure.

$$\begin{cases} A_2 = -\dfrac{A_1}{\sin(a_n l_2) - \sinh(a_n l_2)}\left[Z_{12}\dfrac{\cos(a_n l_2) - \cosh(a_n l_2)}{2} + Z_{11}\sinh(a_n l_2)\right] \\ B_2 = A_1 \dfrac{Z_{12}}{2} \\ C_2 = A_1 \cdot Z_{11} - A_2 \\ A_3 = -\dfrac{A_1}{\sin(a_n l_2) - \sinh(a_n l_2)} \cdot \dfrac{Z_{12}\cdot Z_{21} + Z_{11}\cdot Z_{22}}{Z_{32}} \end{cases} \tag{6}$$

$$\begin{cases} W_1(x_1) = A_1\left[\sin(a_n x_1) - \sinh(a_n x_1) - \dfrac{\sin(a_n l_1) - \sinh(a_n l_1)}{\cos(a_n l_1) - \cosh(a_n l_1)}\left[\cos(a_n x_1) - \cosh(a_n x_1)\right]\right] \\ W_2(x_2) = A_2 \sin(a_n x_2) + B_2\left[\cos(a_n x_2) - \cosh(a_n x_2)\right] + C_2 \sinh(a_n x_2) \\ W_3(x_3) = A_3\left[\sin(a_n x_3) - \dfrac{\sin(a_n l_3)}{\sin(a_n l_3)}\sinh(a_n x_3)\right] \end{cases} \tag{7}$$

with:
$x_1 \in \lfloor 0,\ l_1 \rfloor$; $x_2 \in \lfloor 0,\ l_2 \rfloor$; $x_3 \in \lfloor 0,\ l_3 \rfloor$ and,
constant $A_1$ is chosen so that the mode shape function is normalized ($\pm 1$) for the entire continuous beam.

## 4. ANALYSIS OF THE INFLUENCE OF INTERMEDIATE SUPPORTS ON EIGENVALUES

It is considered that the intermediate supports (fig. 1) can be in any position along the normalized continuous beam ($l_1 + l_2 + l_3 = 1$). It is considered that the intermediate supports (fig. 1) may be placed in any position along the whole length of the normalized continuous beam.
By solving equation (5) for each position of the intermediate supports, in which $l_1$ = (0, 1); $l_2$ = ($l_1$ + *the iteration step); $l_3$ = 1 – $l_1$ - $l_2$*, the eigenvalues for each vibration mode are obtained.
For each vibration mode, the results are integrated in a diagonal matrix whose 3D graphical representation is illustrated in figures 2 – 7 for the first six vibration mode.
The obtained surfaces presented in figures 2 – 7 gives us a general image on the evolution of the eigenvalues depending on the position that the intermediate supports can have.

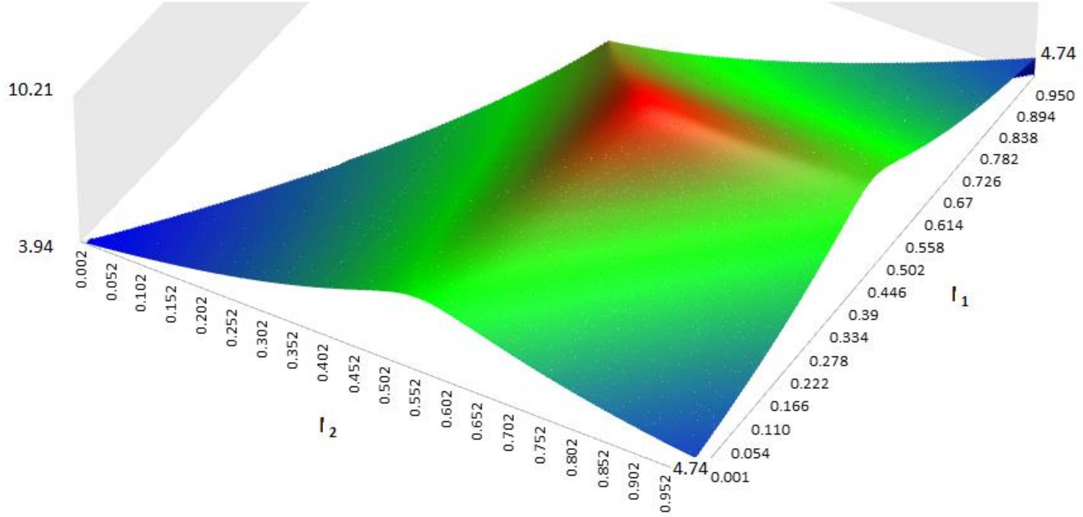


**Figure 2:** Eigenvalues evolution for the 1st vibration mode.

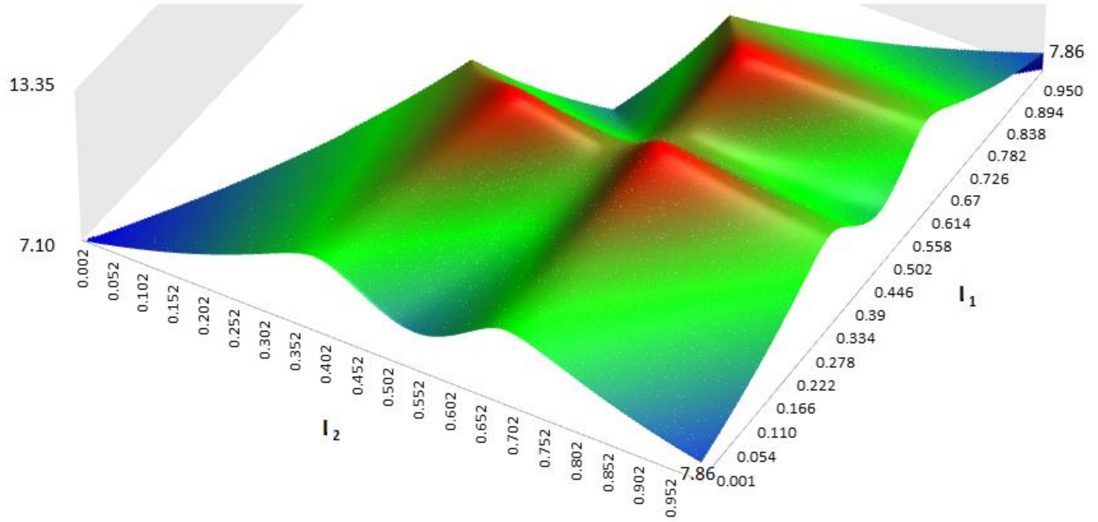


**Figure 3:** Eigenvalues evolution for the 2nd vibration mode.

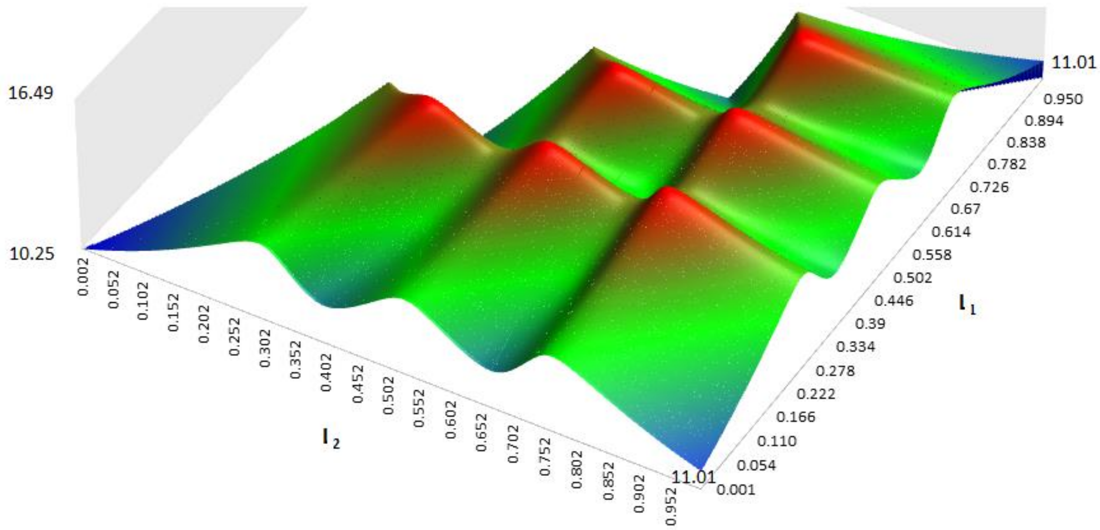


**Figure 4:** Eigenvalues evolution for the 3rd vibration mode.

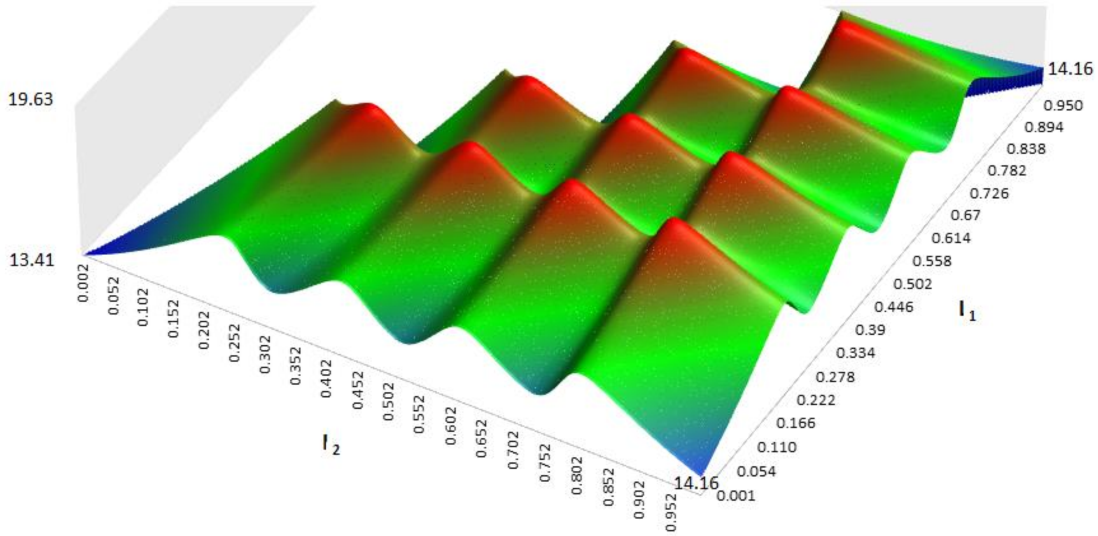


**Figure 5:** Eigenvalues evolution for the 4th vibration mode.

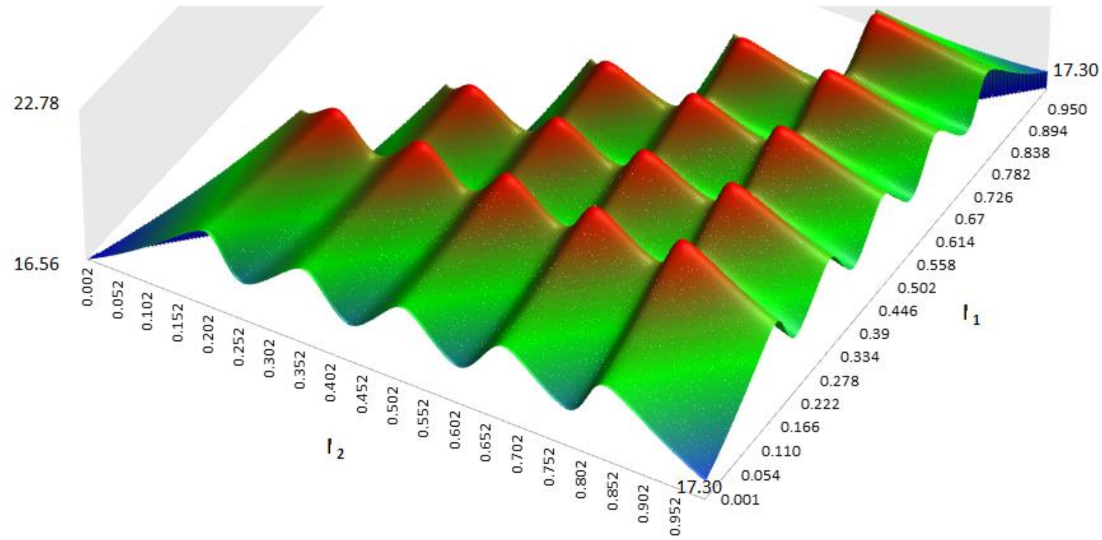


**Figure 6:** Eigenvalues evolution for the 5th vibration mode.

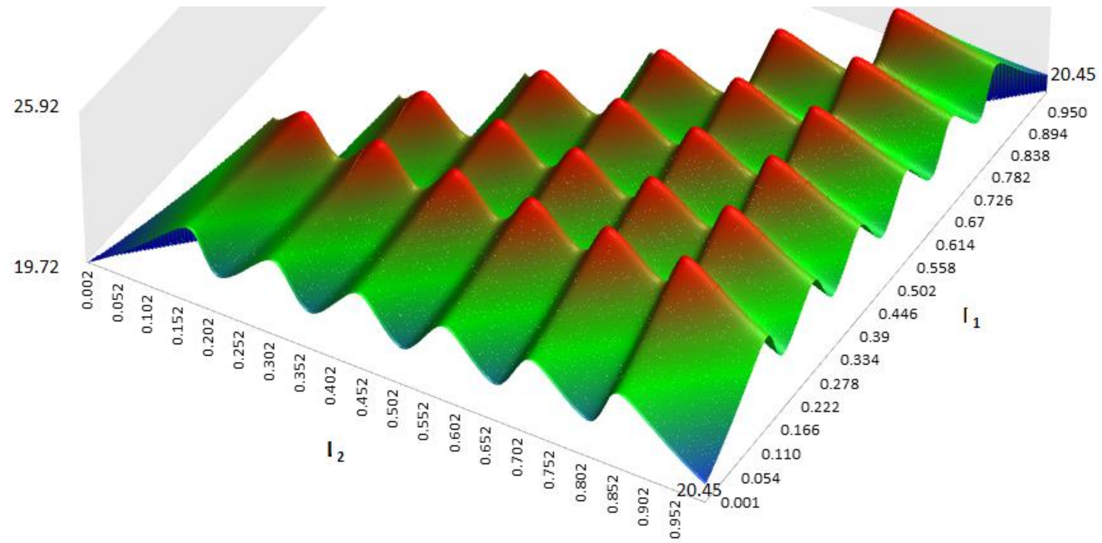


**Figure 7:** Eigenvalues evolution for the 6th vibration mode.

Knowing the eigenvalues for the continuous healthy beam, we can apply relations linked to the transverse crack severity and local bending moment in the slice around the crack and can predict the frequencies of the damaged beam [10]. However, a precise frequency estimation is necessary if damage detection is performed on real structures, thus advance estimation method have to be involved [11].

## 5. CONLUSIONS

The paper presents a generalized relationship (5) for the calculation of eigenvalues for a continuous beam with three spans regardless of the type of support considered. For example, in the present paper the case of the continuous beam clamped at one end and hinged at the other was chosen.

For this analyzed case, the relations that allow the calculation of the integration coefficients (6) are presented, as well as the modal functions that describe the vibration mode shape (7).

The influence of the position of the intermediate supports on the eigenvalues, when they can be placed in any position on the opening of the continuous beam is illustrated by a 3D representation, for the first six modes of vibration.

For particular cases, when $l_1 \to 0$ and $l_2 \to 0$, the continuous beam with three spans behaves like a beam clamped at the left end and hinged at the right end; when $l_1 \to 0$, $l_2 \to 1$, or $l_1 \to 1$, $l_2 \to 1$ the continuous beam with three spans behaves like a beam clamped at both ends.

Knowing the analytical expression of the modal function, it is easy to obtaine the mode shape curvature function, on which depends the establishment of the location of a damage on the beam [8, 9, 10], in case of its appearance.

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