Transilvania University of Brasov
FACULTY OF MECHANICAL ENGINEERING

ICMS 2019 & COMEC 2019
Brasov, ROMANIA, 21-22 November 2019

PSEUDOSPECTRA AND LYAPUNOV STABILITY

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Abstract: *Classical stability analysis of linear models is based upon eigenvalues. This is most notably true for self-adjoint matrices and operators, which possess a basis of orthogonal eigenvectors. In recent decades, recognition has grown that one must proceed with greater caution when a matrix or operator lacks an orthogonal basis of eigenvectors (non-normal operators). In this paper we use Lyapunov equations and functions to consider perturbed matrices. The basic question is: what choice of Lyapunov function V would allow the largest perturbation and still guarantee that dV/dt is negative definite? By using a sub-optimal strategy and pseudospectra we find that this is determined by testing for the existence of solutions to a related 'quadratic' matrix equation - algebraic Riccati equation (ARE).*
Keywords: *perturbations, stability, pseudospectrum, Riccati equations*

1. INTRODUCTION

It is well known that the Lyapunov direct method represents an approach to the problem of stability for both linear and nonlinear dynamical systems, and for a variety of fields. In recent decades, recognition has grown that one must proceed with greater caution when a matrix or operator lacks an orthogonal basis of eigenvectors. Such operators are called non-normal, and this property can lead to a rich variety of behavior. For example, non-normality can be associated with transient behavior that differs entirely from the asymptotic behavior suggested by eigenvalues. Such transients may manifest themselves in slow convergence of iterative processes, in nearness to instability, and in the transition to turbulence in fluid flow.
In this paper we use Lyapunov equations and functions to consider perturbed matrices. We will consider only affine unstructured parameter uncertainty (perturbations).
The basic question is: what choice of Lyapunov function V would allow the largest perturbation and still guarantee that dV/dt is negative definite? By using a sub-optimal strategy and pseudospectra we find that this is determined by testing for the existence of solutions to a related 'quadratic equation with matrix coefficients and unknowns - the so-called matrix Riccati equation.

2. PSEUDOSPECTRA AND PERTURBED MATRICES

We consider a matrix A in the space of n-by-n matrices K^{nxn}, K= R or C.
Suppose that A is uncertain or subjected to a perturbation at the form $\Delta \to A(\Delta)$ where Δ represent a vector or matrix of parameter deviations. In this paper we will consider only affine unstructured parameter perturbations $A(\Delta) = A + \Delta$.
We denote the spectrum of A by $\sigma(A)$ is its set of eigenvalues

$$\sigma(A) = \{\lambda \in C : \det(A - \lambda I) = 0\} = \{point\ where\ (A - \lambda I)^{-1}\ is\ undefinined\} \quad (1)$$

and we denote by $\alpha(A)$ the spectral abscissa of A which is the largest of the real parts of eigenvalues

$$\alpha = \sup\{\operatorname{Re} z : z \in \sigma(A)\} \quad (2)$$

For a real ε > 0, the ε-*pseudospectrum* of A is the set

$$\sigma_\varepsilon(A) = \{z \in C : z \in \sigma(X)\ where \|X - A\| < \varepsilon\}. \quad (3)$$

Throughout, $\|\cdot\|$ denotes the operator 2-norm. Any element of the pseudospectrum is called a *pseudo-eingenvalue.*
The ε-pseudospectral abscissa α_ε is the maximum value of the real part over the pseudo-spectrum:

$\alpha_{\varepsilon} = \sup\{\mathrm{Re}\, z : z \in \sigma_{\varepsilon}\}$. (4)

In this paper we investigate the set of eigenvalues of a perturbed matrix $A + \Delta \in \mathrm{R}^{n x n}$ where A is given and $\Delta \in \mathrm{R}^{n x n}$, $\|\Delta\| < \rho$ is arbitrary.

For any $A \in \mathrm{K}^{n x n}$ we call

$$\sigma(A, \rho) = \{\lambda \in \mathrm{C}; \exists\ \Delta \in \mathrm{R}^{n\ x n} : \|\Delta\| < \rho \ \mathit{and}\ \lambda \in \sigma(A + \Delta)\} \quad (5)$$

the *spectral value set of A with perturbation radius ρ* and

$$d_K = \min\{\|\Delta\| : \Delta \in \mathrm{K}^{n x n}, \sigma(A + \Delta) \cap \mathrm{C}_{+} \neq \emptyset\} \quad (6)$$

the *distance from instability* [1], i.e. is the distance, within the normed space $(\mathrm{K}^{n x n}, \|\cdot\|)$, between A and the set of C - unstable matrices in K^{nxn} .

For a normal matrix A the distance from instability is measured by the distance of its spectrum from the imaginary axis. If A is not normal, the distance of $\sigma(A)$ from the imaginary axis can be a very misleading indicator of the robustness of stability of A.

Just as the spectral abscissa of a matrix provides a measure of its stability, so the ε-pseudospectral abscissa provides a measure of robust stability, where by robust we mean with respect to complex perturbations in the matrix [5], [6], [7], [8].

The pseudo-abscissa $\alpha_{\varepsilon}(A)$ is the maximum value of the real parts over the pseudo-eigenvalues

$$\alpha_{\varepsilon}(A) = \sup\{\mathrm{Re}\, z : z \in \sigma_{\varepsilon}(A)\},\ \alpha_0(A) = \alpha(A) \quad (7)$$

where $\sigma_{\varepsilon}(A)$ are the ε-pseudospectra of matrix A.

The ε-pseudospectra of A consist of all eigenvalues of matrices within a distance ε of A, and in particular, $\Lambda_0(A)$ is just the spectrum of A. Pseudospectra plot the set of eigenvalues of A+Δ for all Δ with $\|\Delta\| < \varepsilon$. We can define $\Lambda_{\varepsilon}(A)$ in various equivalent ways: [1], [2], [3], [4], [5], [6], [8].

$$\Lambda_{\varepsilon} = \left\{ z \in C : \left\|(zI - A)^{-1}\right\| \geq \varepsilon^{-1} \right\}$$

$$\Lambda_{\varepsilon} = \left\{ z \in C : z \in \Lambda(A + \Delta)\ \mathit{for\ some}\ \Delta\ \mathit{with}\ \|\Delta\| \leq \varepsilon \right\}.$$

3. LYAPUNOV EQUATION AND UNSTRUCTURED PERTURBATIONS

We consider a linear time invariant finite dimensional systems of the form

$$\dot{x}(t) = Ax(t),\ t \in \mathrm{R}_{+};\quad (x(t+1) = Ax(t),\ t \in \mathrm{N}\} \quad (8)$$

It is well known that the linear dynamical system described by equation (7) is asymptotically stable if

$$\sigma(A) \subset \mathrm{C}_{-} = \{z \in \mathrm{C}\cdot; \mathrm{Re}\, z < 0\};\ \text{or}\ \sigma(A) \subset \mathrm{C}_1 = \{z \in \mathrm{C}\cdot; |z| < 1\}\ . \quad (9)$$

equivalently, for any positive definite matrix Q we can find a positive definite matrix P which satisfies the matrix Lyapunov equation

$$A^T P + AP = -Q\ . \quad (10)$$

We need to consider not just one nominal system (8) but a family of models

$$\dot{x} = (A + \Delta)x \quad (11)$$

where Δ is a perturbation.

The fundamental question is how large can we allow ρ so if $\|\Delta\| < \rho$ then all eigenvalues of perturbed matrix A+ Δ are guaranteed to have negative real part.

Let λ be an eigenvalue of A+Δ. Then

$$(A + \Delta)\, u = \lambda\ u \Leftrightarrow (\lambda\ I - A)\, u = \Delta u \Leftrightarrow u = (\lambda\ I - A)^{-1} \Delta u\ .$$

Tacking magnitude of both sides and using matrix norm we obtain

$$\left\|(\lambda I - A)^{-1}\right\| \|\Delta\| \geq 1,\ \text{but}\ \|\Delta\| < \rho \ \Rightarrow\ \left\|(\lambda\ I - A)^{-1}\right\| > \frac{1}{\rho} \quad (12).$$

So, the pseudospectrum crosses over the imaginary axis at a value of ρ and an imaginary number $z = i\,y$ if

$$\left\|(iy I - A)^{-1}\right\| \geq \frac{1}{\rho}.$$

It is know that A is stable, then

$$A^T P + AP = -Q$$
has a positive definite solution.

Let $V(x) = x^T P x$ be Lyapunov function. The $\frac{dV}{dx}$ calculation for perturbed system (11), with P solution of (7), gives

$$\frac{dV}{dx} \leq -\lambda_1 |x|^2 + 2\|P\|\,\|\Delta\|\,|x|^2 \qquad (13)$$

where $0 < \lambda_1 \leq \lambda_2 \leq ....\lambda_n$ are the positive eigenvalues of the positive definite matrix Q.

From (13) result that $\frac{dV}{dx}$ will be negative definite, from given Q, if we chose

$$\rho = \frac{\lambda_1}{2\|P\|} \qquad (14).$$

So for ρ given by (14) and for a given Q, if $\|\Delta\| < \rho$, then every eigenvalue of $(A + \Delta)$ has negative real part.

4. CHARACTERIZATION OF STABILITY RADIUS VIA A PARAMETRIZED RICATTI EQUATION

In this section we study a strategy for increasing the size of ρ, i.e. choose Q to maximize ρ given by (12). It is obvious that the range of ρ is dependent of choose of Q. To see this dependence we look at the eigenvalues of (A+ Δ) directly. For a given value of ρ > 0 we look to plot the totality or set of eigenvalues of (A+Δ) for all Δ with $\|\Delta\| < \rho$. For ρ very small we would expect this totality or set of eigenvalues to be focused simply around the eigenvalues of the nominal A. As we increase ρ then the set grows, expanding in sometimes amazingly pretty shapes. A question arise: can we get a handle on the smallest value of ρ so that the contours intersect the right-half plane especially by our Lyapunov Equation technique. The smallest ρ is determined as fallows:

Theorem 1. Every eigenvalue of (A+ Δ), with $\|\Delta\| < \rho$ has negative real part if and only if the Riccati equation

$$A^T P + PA + \rho^2 I + P^2 = 0 \qquad (\text{ARE}_\varepsilon)$$

has a positive definite solution.

The above theorem can be used to construct a Lyapunov function of maximal robustness. Indeed if P is any solution of (ARE$_\rho$) with $\rho = d_K$ then it can be shown that $V(x) = x^T P x$ is a Lyapunov function for the set of perturbed systems $\dot{x} = (A + \Delta)x$, $\|\Delta\| < \rho$, $\rho \in [0, d_K(A))$.

Proof.

$(A+\Delta)^T P + P(A+\Delta) = -\rho^2 I - P^2 + \Delta^T P + P\Delta = -(P+\Delta)^T (P+\Delta) + \Delta^T \Delta - \rho^2 I \equiv -Q$ provided that $\|\Delta\| < \rho$.

From (12) we shown that the smallest ρ for which this can happen is the smallest we can make $\frac{1}{\left\|(iyI - A)^{-1}\right\|}$ ($\rho_{\min}$). Now why this smallest $\rho_{\min}$ got anything to do with the (ARE$_\rho$)? If this equation has a solution then we can write

$$(iyI - A)^* P + P(iyI - A) - \rho^2 I - P^2 = 0$$

Here $(\cdot)^*$ indicates transpose and complex conjugate. Then, multiplying the equation (on the left) by inverses of $(iyI - A)^*$ and $(iyI - A)$ resp., we obtain

$$I - \rho^2 (iyI - A)^{-1*} (iyI - A)^{-1} = (P(iyI - A)^{-1} - I)^* (P(iyI - A)^{-1} - I)$$

It follows from this that necessarily $\rho \leq \frac{1}{\left\|(iyI - A)^{-1}\right\|}$.

So we prove that the pseudospectrum at level ρ lies entirely in left-half complex plane if and only if the Riccati equation (ARE$_\varepsilon$) has a positive definite solution.

5. CONCLUSIONS

The above theorem can be used to construct a Lyapunov function of maximal robustness. Indeed if P is any solution of (ARE$_\varepsilon$) with $\varepsilon = d_K$ then it can be shown that $V(x) = x^T P x$ is a Lyapunov function for the set of perturbed systems $\dot{x} = (A + \Delta)x$, $\|\Delta\| < \varepsilon$, $\varepsilon \in [0, d_K(A)]$.

Pseudospectra is a new computational and visualization tool, which is making significant impacts across stability and control of dynamical systems. This paper is just meant to motivate more research in finding applications in robustness of stability and controllability of dynamical systems.

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