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PSEUDOSPECTRA AND DYNAMICAL SYSTEMS APPLICATIONS

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Abstract: *This paper is devoted to describing and illustrating pseudospectra and the connections of pseudospectra with related quantities for non-normal matrices, in particular, the distance to instability, the H_∞ norm, and distance to uncontrollability. Pseudospectra and related quantities became a standard tool in the 1990s, with applications in mechanics, numerical analysis, operator theory, control theory, differential and integral equation. In all of this fields it has been found that in case of pronounced non-normality, eigenvalues and eigenvectors alone do not always reveal much about the aspects of the behavior of a matrix or operator that matter in applications, including phenomena of stability, convergence, and resonance, and that pseudo-eigenvalues and pseudo-eigenvector may do better.*
Keywords: *pseudospectra, perturbed dynamical systems, Lyapunov stability*

1. INTRODUCTION

A new tool has become popular in the 1990s for the study of matrices and linear operators. The traditional tool is eigenvalues or *spectra* for matrices or linear operators respectively. Eigenvalues and spectra tend to be less informative however when the matrix or operator is non - Hermitian or more generally non-normal. For some matrices, non-normality (nonorthogonality of the eigenvectors) may be physically important. For example, non-normality can be associated with transient behavior that differs entirely from the asymptotic behavior suggested by eigenvalues. In extreme cases, non-normality combined with the practical limits of machine precision can lead to difficulties in accurately finding the eigenvalues.
Pseudospectra is a familiar tool available for learning more about the cases in which non-normality may be important. Pseudospectra are sets in the complex plane bounded by level curves of the norm of the resolvent.
Pseudospectra seem to have been invented independently (with different names) at least five times by Henry Landau (1975,1976,1977), Varah (1979), Godunov, Kiriljuk and Kostin (1990), and Hinrichsen, Pritchard and Kelb (1992,1993). The first person to define the notion of pseudospectra was Henry Landau at Bell Laboratories in the 1970s, who was motivated in part by applications in lasers and optical resonator. Figure 1 shows pseudospectra of integral operator investigated by Landau [6].
Pseudospectra and related quantities for non-normal matrices and operators where investigated in the 1970s and 1980s [3] and became a standard tool in the 1990s [11], with applications in fluid mechanics, numerical analysis, operator theory, control theory [4], differential and integral equation [2]. In all of this fields it has been found that in case of pronounced non-normality, eigenvalues and eigenvectors alone do not always reveal much about the aspects of the behavior of a matrix or operator that matter in applications, including phenomena of stability, convergence, and resonance, and that pseudo-eigenvalues and pseudo-eigenvector may do better.

2. SPECTRA AND PSEUDOSPECTRA. COMPUTATIONAL ASPECT

Suppose we have a square *n*-by-*n* complex matrix, $A \in C^{n\times n}$. The *spectrum* of *A* denoted by $\Lambda(A)$, is it set of eigenvalues, a finite subset of *C* consisting of a most *n* points. For any $z \in C$ the *resolvent* of *A* at *z* is the matrix or linear operator $(zI - A)^{-1}$, if this exists. An equivalent definition of $\Lambda(A)$ is that it is the set of $z \in C$ where the resolvent does not exist or is unbounded.
So, the eigenvalues of *A* satisfy the following definition

$$\Lambda(A)=\{\lambda\in C:\det(A-\lambda I)=0\}=\{\text{point where}(A-\lambda I)^{-1}\text{is undefinined}\} \quad (1)$$

If λ is an eigenvalue of A, then by convention we define the norm of $\left\|(A-\lambda I)^{-1}\right\|$ to be infinity. But what if $\left\|(A-\lambda I)^{-1}\right\|$ is finite but very large?

This pattern of thinking leads to a first definition of ε-pseudospectrum

$$\Lambda_{\varepsilon}(A)=\{z\in C:\left\|(A-zI)^{-1}\right\|\geq\varepsilon^{-1}\} \quad (2)$$

Note $\Lambda_0(A)=\Lambda(A)$. In words the pseudospectrum is the subset of the complex plane bounded by the ε^{-1} level curve or curves of the resolvent norm. Equivalently, the ε-pseudospectrum can be defined in terms of eigenvalues of perturbed matrices

$$\Lambda_{\varepsilon}(A)=\{z\in C:z\in\Lambda(A+\Delta),\ \|\Delta\|\leq\varepsilon\} \quad (3)$$

Here Δ is a perturbation. In words the pseudospectrum is the set of all complex numbers that are in the spectrum of some matrix or operator obtained by a perturbation of norm at most ε. This definition implies that pseudospectra can be interpreted in terms of perturbations of spectra but this does not mean that the analysis of perturbations is the main thing pseudospectra are useful for.

On the contrary other aspects of behavior of a matrix or linear operator tend to be more important in applications including growth or decay of $\|A^n\|$ as a function of *n* and growth or decay of $\|e^{tA}\|$ as a function of *t*.

A starting point for computations is a third equivalent defintion of pseudospectra. If $\sigma_{\min}(A)$ denotes the smallest singular value of *A* then we have

$$\Lambda_{\varepsilon}(A)=\{z\in C:\sigma_{\min}(zI-A)\leq\varepsilon\} \quad (4)$$

Thus the pseudospectra of *A* are the sets in the *z*-plane bounded by level curves of the function $\sigma_{\min}(zI-A)$.

For details of the equivalence of (2-4) see for example van Dorsselaer, Kraaijevanger and Spijker (1993).The mathematical foundations of such material are set forth in the book by Kato (1976) [7].

If the matrix or operator *A* is normal (i.e., it has an orthogonal basis of eigenvectors), then its 2-norm epsilon-pseudospectrum consists of closed balls of radius epsilon surrounding the eigenvalues, Figure 2.

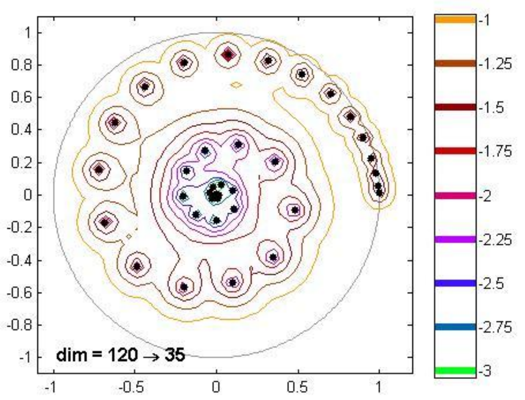

Fig. 1 ε-pseudospectrum of Landau operator

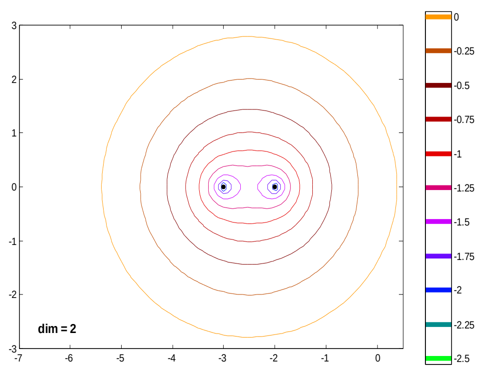

Fig. 2 2-norm ε-pseudospectrum for a normal matrix

All pseudospectra plots follow this general template. The eingesvalues are plotted as black dots on the complex plane, and colored lines mark the boundaries of various pseudospectra. The color bar on the right indicates the log10 of each boundnary. Notice that for some values of epsilon the pseudospectrum is connected, while for smaller epsilon it can consist of disjoint sets. Sometimes the pseudospectral boundary about an eigenvalue is too small to be clearly visible on the plots.

For calculate pseudospectra properly, the place to begin is with the singular value decomposition, tacking advantage of definition (4). The obvious algorithm is to evaluate $\sigma_{\min}(zI - A)$ for values of z on a grid in the complex plane and then generate a contour plot from this data, for example in MATLAB code. Computing the pseudospectra of a matrix of dimension N is traditionally an expensive task, requiring an $O(N^3)$ singular value decomposition at each point in a grid. The reason for the cost of computation of pseudospectra is now clear: the amount of work needed to compute the minimum singular value of a general matrix of dimension N is $O(N^3)$. However, several techniques have been developed to reduce this cost [7].
A MATLAB code (Matlab GUI) for computing pseudospectra, created by Tom. Wright & Nick Trefethen, is available at www.comlab.ox.ac.uk/ouc1/people/nick.trefethen.html. **M**atlab **G**raphical user **I**nterface (GUI), which automates the computation of pseudospectra after the eigenvalues of a matrix have been computed by eigs in Matlab. Figure 3 shows a snapshot after a run of EigTool. Initially the pseudospectra are computed on a coarse grid to give a fast indication of the nonnormality of the matrix, but the GUI allows control over the number of grid points if a higher quality picture is desired. Other features include the abilities to change the contour levels shown without recomputing the underlying values, and to select parts of the complex plane to zoom in for greater detail. The GUI can also be used as a graphical front end to our other pseudospectra codes for computing pseudospectra of smaller general matrices.
For nonsymmetric matrices, the mathematical basis of these packages is the Arnoldi iteration with implicit restarting [8], which works by compressing the matrix to an “interesting” Hessenberg matrix, one which contains information about the eigenvalues and eigenvectors of interest. For general information on large-scale nonsymmetric matrix eigenvalue iterations, see [3].

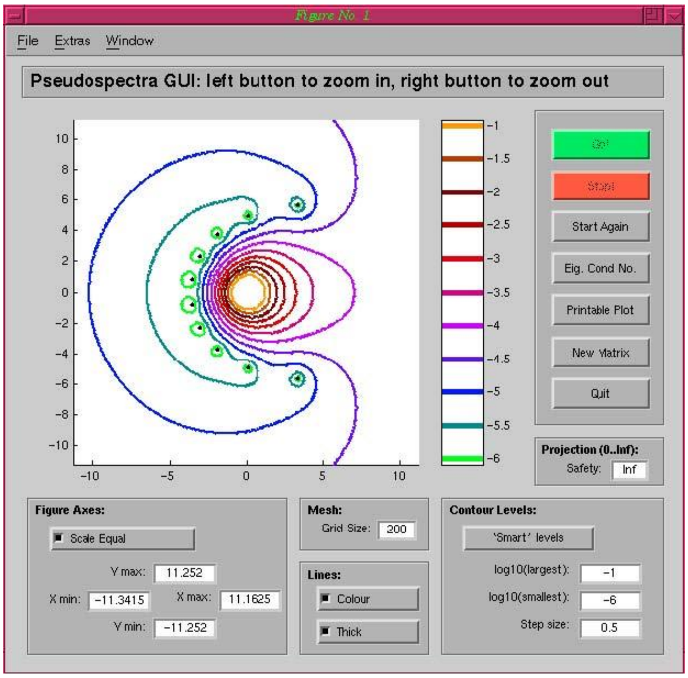

Figure 3: a snapshot after a run of EigTool

3. UNSTRUCTURED PERTURBATION AND LYAPUNOV EQUATION TECHNIQUE

In section we use Lyapunov equations and functions to consider perturbed matrices. We will consider only affine unstructured parameter uncertainty (perturbations).
The basic question is: what choice of Lyapunov function V would allow the largest perturbation and still guarantee that dV/dt is negative definite? By using a sub-optimal strategy and pseudospectra we find that this is determined by testing for the existence of solutions to a related ‘quadratic’ equation with matrix coefficients and unknowns - the so-called matrix Riccati equation.
We consider a linear time invariant finite dimensional systems of the form

$$\dot{x}(t) = Ax(t),\ t \in \mathbb{R}_+; \quad (x(t+1) = Ax(t),\ t \in \mathbb{N}\} \tag{5}$$

It is well known that the linear dynamical system described by equation (5) is asymptotically stable if

$\Lambda(A) \subset \mathbb{C}_- = \{z \in \mathbb{C};\ \operatorname{Re} z < 0\}$; or

$\Lambda(A) \subset \{z \in \mathbb{C};\ |z| < 1\}$; (6)

Equivalently, for any positive definite matrix Q we can find a positive definite matrix P which satisfies the matrix Lyapunov equation

$$A^T P + AP = -Q \tag{7}$$

We need to consider not just one nominal system (6) but a family of models

$$\dot{x} = (A + \Delta)\, x \tag{8}$$

where Δ is a perturbation.

The fundamental question is how large can we allow ε so if $\|\Delta\| < \varepsilon$ then all eigenvalues of perturbed matrix A+ Δ are guaranteed to have negative real part.

Let λ be an eigenvalue of A+Δ. Then

$$(A + \Delta)\, u = \lambda\ u \Leftrightarrow (\lambda\ I - A)\, u = \Delta u \Leftrightarrow u = (\lambda\ I - A)^{-1} \Delta u\, .$$

Tacking magnitude of both sides and using matrix norm we obtain

$$\|(\lambda I - A)^{-1}\|\ \|\Delta\| \geq 1 \text{ but } \|\Delta\| < \varepsilon \ \Rightarrow\ \|(\lambda\ I - A)^{-1}\| > \frac{1}{\varepsilon} \tag{9}$$

So, the pseudospectrum crosses over the imaginary axis at a value of ε and an imaginary number $z = i\, y$ if $\|(iy I - A)^{-1}\| \geq \dfrac{1}{\varepsilon}$.

Let $V(x) = x^T P x$ be Lyapunov function. The $\dfrac{dV}{dx}$ calculation for perturbed system (8), with P solution of (7), gives

$$\frac{dV}{dx} \leq -\lambda_1 |x|^2 + 2\|P\|\ \|\Delta\|\ |x|^2 \tag{10}$$

where $0 < \lambda_1 \leq \lambda_2 \leq ....\lambda_n$ are the eigenvalues of the positive definite matrix Q.

From (10) result that $\dfrac{dV}{dx}$ will be negative definite, from given Q, if we chose

$$\varepsilon = \frac{\lambda_1}{2\|P\|} \tag{11}$$

So for ε given by (11) and for a given Q, if $\|\Delta\| < \varepsilon$, then every eigenvalue of (A+ Δ) has negative real part.

4. STABILITY RADIUS VIA PARAMETRIZED RICCATI EQUATION

In this section we study a strategy for increasing the size of ε, i.e. choose Q to maximize ε given by (13). It is obvious that the range of ε is dependent of choose of Q. To see this dependence we look at the eigenvalues of (A+ Δ) directly. For a given value of ε > 0 we look to plot the totality or set of eigenvalues of (A+Δ) for all Δ with $\|\Delta\| < \varepsilon$. For ε very small we would expect this totality or set of eigenvalues to be focused simply around the eigenvalues of the nominal A. As we increase ε then the set grows, expanding in sometimes amazingly pretty shapes.

A question arise: can we get a handle on the smallest value of ε so that the contours intersect the right-half plane especially by our Lyapunov Equation technique. The smallest ε is determined as fallows:

Theorem. Every eigenvalue of (A+Δ), with $\|\Delta\| < \varepsilon$ has negative real part if and only if the parametrized algebraic Riccati equation

$$A^T P + PA + \varepsilon^2 I + P^2 = 0 \qquad (\text{ARE}_\varepsilon)$$

has a positive definite solution.
P r o o f.

$$(A+\Delta)^T P + P(A+\Delta) = -\varepsilon^2 I - P^2 + \Delta^T P + P\Delta = -(P+\Delta)^T (P+\Delta) + \Delta^T \Delta - \varepsilon^2 I \equiv -Q$$

provided that $\|\Delta\| < \varepsilon$. From (9) we shown that the smallest ε for which this can happen is the smallest we can make $\dfrac{1}{\|(iyI-A)^{-1}\|}$ ($\varepsilon_{\min}$).

Now why this smallest $\varepsilon_{\min}$ got anything to do with the (ARE$_\varepsilon$)? . If this equation has a solution then we can write

$$(iyI-A)^* P + P(iyI-A) - \varepsilon^2 I - P^2 = 0\,.$$

Here $(\cdot)^*$ indicates transpose and complex conjugate. Then, multiplying the equation (on the left) by inverses of $(iyI-A)^*$ and $(iyI-A)$ resp., we obtain

$$I - \varepsilon^2 (iyI-A)^{-1*} (iyI-A)^{-1} = (P(iyI-A)^{-1} - I)^* (P(iyI-A)^{-1} - I)\,.$$

It follows from this that necessarily $\varepsilon \le \dfrac{1}{\|(iyI-A)^{-1}\|}$.

So we prove that the pseudospectrum at level ε lies entirely in left-half complex plane if and only if the Riccati equation (ARE$_\varepsilon$) has a positive definite solution.

5. PSEDOSPECTRA AND CONTROL OF DYNAMICAL SYSTEMS

For real matrices *A* and *B* of size $n \times n$ and $n \times m$ respectively, consider the control system defined by

$$\dot{x} = Ax + Bu \qquad (12)$$

This system is said be state controllable at $t = t_0$ if there exist a piecewise continuous input $\boldsymbol{u}(t)$ that will drive the initial state $\boldsymbol{x}(t_0)$ to any final state $\boldsymbol{x}(t_f)$.
Classical theory provides a simple characterization of controllability. The above system is controllable exactly when the matrix $[A-zI \mid B]$ has full row rank, *n*, for all scalars $z \in$ C.
A small distance to uncontrollability correlates with various difficulties for the control system, including numerical challenges for associated "pole placement" problems. A simple argument [1] shows that the distance to uncontrollability is given by

$$\min_{z\in\mathbf{C}} \sigma_{\min}[A - zI \mid B]\,. \qquad (13)$$

The function to be minimized in the expression (12) has lower level sets of the form

$$\{z \in \mathbf{C} : \sigma_{\min}[A - zI \mid B] \le \varepsilon\} \qquad (14)$$

which is the pseudospectra, when matrix *B* is empty. Substantial insight is gained from examples, so consider the parameterized matrix pair, [1],

$$(A,B) = \left[\begin{bmatrix} 1 & 1 & 2 & 3 \\ -1 & 1 & 4 & 5 \\ 0 & \eta_1 & 1 & 2 \\ \eta_2 & 0 & -2 & 1 \end{bmatrix} \middle| \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}\right], \qquad (15)$$

where η_1 and η_2 are real parameters. Figures 4 and 5 show pseudospectra (produced using by T.Wright's software EigTool [5]) for, respectively, the controllable pair (15) when $\eta_1 = \eta_2 = 1$ and the un controllable pair (15) when $\eta_1 = \eta_2 = 0$.

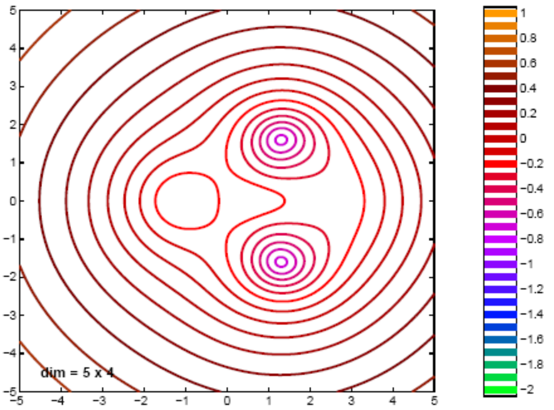

Figure 4: Pseudospectra for the controllable pair (14) with $\eta_1 = \eta_2 = 1$

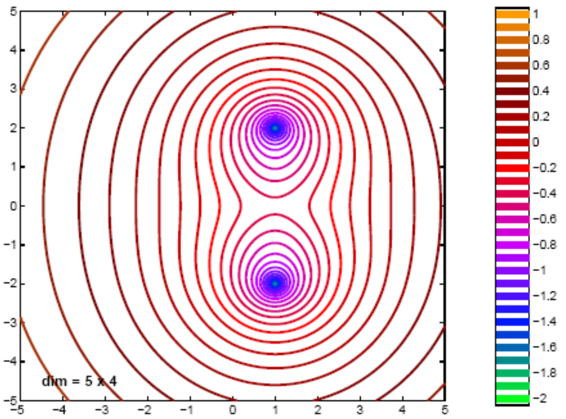

Figure 5: Pseudospectra for the uncontrollable pair (14) with $\eta_1 = \eta_2 = 0$

The horizontal and vertical axes in the figures show the real and imaginary parts of *z*. The legends on the right sides of the figures show the contour heights (values of ε) on a log 10 scale, with both plots using the same scale. In Figure 4, the "pseudospectral landscape" has three local minima and one can estimate that the global minimum value (by definition, the distance to uncontrollability) is about $10^{-0.75}$ (in fact, it is 0.1872). In Figure 5, there are only two local minima (forming a complex conjugate pair), and one can see that the contours drop too much lower values (in fact, it is easy to check that the minimal value of (1.1) is zero at the points $z = 1 \pm 2i$).
In Figure 4, the pseudospectra contain up to, but no more than, three connected components, depending on the choice of ε, while the pseudospectra in Figure 5contain up to, but no more than, two connected components. Maximization of the distance to uncontrollability for smoothly varying parameterized pair $(A, B)(x)$ over vector of free parameters $x \in \mathbf{R}^k$, is given with two algorithms namely the Trisection Variant of Gu's Algorithm and the BFGS/Gu Hybrid, [1]. Matlab implementations of the algorithms are freely available: http://www.cs.nyu.edu/faculty/overton/software.

6. CONCLUSION

Pseudospectra and related quantities for matrices (non-normal, dense, and sparse) and operators became a standard tool, with application in mechanics, optimization, control theory, numerical analysis, differential and integral equations. In all of these fields it has been found that in cases of pronounced non-normality, eigenvalues and eingenvectors alone do not always reveal much about the aspects of the behavior of a matrix or operators that matter in applications, including phenomena of stability, controllability, convergence, and that pseudo-eigenvalues and pseudo-eingenvectors may batter.
This paper is devoted to describing and illustrating pseudospectra and the connections of pseudospectra with related quantities, in particular, the distance to instability, the $\mathbf{H}_\infty$ norm, and distance to uncontrollability.

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