The 4th International Conference
"Computational Mechanics
and Virtual Engineering"
COMEC 2011
20-22 OCTOBER 2011, Brasov, Romania

ON ANALYTICAL AND EXPERIMENTAL SOLUTIONS FOR LARGE DEFLECTIONS OF CLAMPED CIRCULAR PLATE

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Abstract: *In this paper, the analytical land experimental solutions for large deflections of clamped circular diaphragm with built – in stress and finite element analysis are presented. The analytical solution is compared to both finite element and experimental results from a clamped circular diaphragm. The large deflections of a clamped circular plate under initial tension simulating a typical micro pressure sensor are studied. The approach follows von Karman plate theory for large deflections and incorporates with a finite difference numerical scheme, for solving the coupled nonlinear differential equations for transverse deflection and force resultants. A uniformly distributed transverse pressure is considered and the linear version of the problem concerning small deflection is investigated first as a preliminary study.*
Keywords: *large deflection, circular plate, finite element, experimental method.*

1. INTRODUCTION

Structural elements in the form of thin circular plates are used in practice in the form of caps, bottoms of tanks and pistons, window access, pressure sensor membrane. Calculation of stress and displacement in such plates can be made with Kirchhoff's theory. The Kirchhoff's linear plate bending theory is valid only for small deflections (deflection is greater and equal to 0,2 from thickness h, $w \le 0{,}2h$) [1, 4, 10, 12]. The linear theory ignores straining of the middle surface of the plate and the corresponding in-plane stresses are neglected. However, if the magnitude of the lateral deflections increases beyond a certain level ($w > 0{,}3h$), these deflections are accompanied by stretching of the middle surface. As the ratio w/h increases, the role of the membrane forces becomes more pronounced. When the magnitude of the maximum deflection reaches the order of the plate thickness ($w \approx h$), the action on the membrane becomes comparable to that of bending. In certain cases, the large displacement problem can be solved analytically [2, 3, 6, 8, 9, 10, 11, 12, 14, 17] but in most of the cases it can be solved numerically [6, 8, 10]. In the paper [7], a simple and efficient incremental load method is developed for bending analysis of thin circular plates with large deflection. A simply supported thin circular plate under uniform loads is considered in paper [7] as an example to illustrate the application of the developed method. The paper [9] is concerned with the bending analysis of an axisymmetric simply clamped circular thin plate with large displacements, so it presents an analytical [10] and an experimental displacement approach for the models subjected to axial symmetrical loads. The problem of large deflection of a clamped circular plate under uniform pressure is studied by the method of successive approximation in terms of the parameter representing the ratio of the center deflection to the thickness [17]. The large array of available methods requires a unitary approach to the problem. In this paper, the small deflection characteristics and the transitions from pure plate to pure membrane behavior are detailed in 2.1. Next, try to present analytical and experimental measurements in thin bent plates with large displacements.

2. MATHEMATICAL FORMULATION OF PROBLEM

Figure 1 shows a circular plate of radius R and thickness h, under an initial in-plane tension load, $N_r = N_0$ and a uniform transverse load $p_z = p_0$. The equilibrium equations for the symmetrical bending of this plate [9] are

$$\frac{dN_r}{dr} + \frac{N_r - N_\theta}{r} = 0 \quad (1) \qquad \frac{d}{dr}\left(rV_r\right) + \frac{d}{dr}\left(rN_r \frac{dw}{dr}\right) + rp_0 = 0 \quad (2) \qquad V_r = \frac{dM_r}{dr} + \frac{M_r - M_\theta}{r} \quad (3)$$

where N_r, N_θ are the lateral loads , V_r is the shear force, M_r, M_θ are the bending moments per unit length, and w is the defection of the plate in the z-direction.

The radial and tangential mid-plane strains assuming von Kármán plate theory for large plate defections are

$$\varepsilon_r = \frac{du}{dr} + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 \qquad \text{and} \qquad \varepsilon_\theta = \frac{u}{r} \tag{4}$$

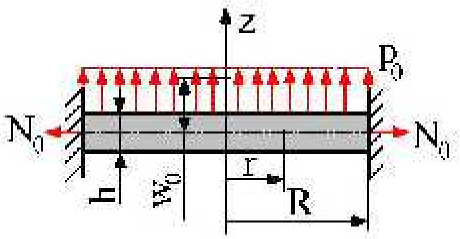

Figure 1. Thin circular plate

The lateral loads and moments per unit length for large plate defections are

$$N_r = B(\varepsilon_r + \nu\varepsilon_\theta); \qquad N_\theta = B(\nu\varepsilon_r + \varepsilon_\theta) \tag{5}$$

and

$$M_r = -D\left(\frac{d^2w}{dr^2} + \nu\frac{1}{r}\frac{dw}{dr}\right); \qquad M_\theta = -D\left(\nu\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) \tag{6}$$

where $B = Eh/\left(1-\nu^2\right)$ and $D = Eh^3/12\left(1-\nu^2\right)$ are the plate extensional and bending stiffness respectively, ν is Poisson's ratio, and E is the modulus of elasticity. Shearing deformations are neglected for the thin plates considered here ($h/r < 1/25$ [8]).

These equations are reduced by first integrating (2),

$$V_r + N_r\frac{dw}{dr} + \frac{p_0 r}{2} = 0 \tag{7}$$

Substituting (6) into the moment equilibrium relation (3) gives

$$V_r = -D\left(\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}\right) \tag{8}$$

Placing equation (6) into (7) produces

$$\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr} - \frac{N_r}{D}\frac{dw}{dr} = \frac{p_0 r}{2D} \tag{9}$$

Rewriting (4) in compatibility form

$$r\frac{d\varepsilon_\theta}{dr} + \varepsilon_\theta - \varepsilon_r + \frac{1}{2}\left(\frac{dw}{dr}\right)^2 = 0 \tag{10}$$

And the inversion of equation (5) yields

$$\varepsilon_r = \frac{1}{Eh}\left(N_r - \nu N_\theta\right) \qquad \text{and} \qquad \varepsilon_\theta = \frac{1}{Eh}\left(-\nu N_\theta + N_r\right) \tag{11}$$

Substituting (11) into the compatibility relation (10) yields

$$\frac{d\left(N_\theta + N_r\right)}{dr} + \frac{Eh}{2r}\left(\frac{dw}{dr}\right)^2 = 0 \tag{12}$$

Introducing (12) into (1) gives

$$\frac{dN_\theta}{dr} - \frac{N_r - N_\theta}{r} + \frac{Eh}{2r}\left(\frac{dw}{dr}\right)^2 = 0 \tag{13}$$

Equations (1), (9), and (13) represent three nonlinear equations in the three unknowns dw/dr, N_r, and N_θ. The nonlinearity appears in the $N_r dw/dr$ term in (9) and the $(dw/dr)^2$ term in (13).

First the circular plate (fig.1) is stretched by an in-plane tension load $N_r = N_0$ around its circumference. The solution of the initial in-plane tension problem is obtained from the general equations (1 - 6) by setting $w = 0$ and $p_0 = 0$. This yields the following results,

$N_r = N_\theta = N_0$; (14) $u = (1-\nu)\frac{N_0 r}{Eh}$ (15) and $\varepsilon_r = \frac{\sigma_r}{E} = \frac{N_r}{Eh}$ (16)

After being initially stretched by the load N_0, the plate is then subjected to the vertical load, p_0. For this case, the lateral loads are decomposed as follows:

$$N_r = N_0 + \widetilde{N}_r ; \qquad (17) \qquad\qquad N_\theta = N_0 + \widetilde{N}_\theta \qquad (18)$$

where $\widetilde{N}_r$ and $\widetilde{N}_\theta$ are incremental changes from N_0 , which are functions of r. The placement of these expressions into (1), (9), (13) yields,

$$\frac{d\widetilde{N}_r}{dr} - \frac{\widetilde{N}_r - \widetilde{N}_\theta}{r} = 0 ; \qquad (19)$$

$$\frac{d\widetilde{N}_\theta}{dr} - \frac{\widetilde{N}_r - \widetilde{N}_\theta}{r} + \frac{Eh}{2r}\left(\frac{dw}{dr}\right)^2 = 0 ; \qquad (20)$$

$$\frac{d^3 w}{dr^3} + \frac{1}{r}\frac{d^2 w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr} - \frac{N_0}{D}\frac{dw}{dr} - \frac{\widetilde{N}_r}{D}\frac{dw}{dr} = \frac{p_0 r}{2D} . \qquad (21)$$

We define the following non-dimensional quantities

$$W = \frac{w}{h};\ U = \frac{u}{h};\ x = \frac{r}{R};\ \theta = \frac{dw}{dx} = \frac{R}{h}\frac{dw}{dr};\ \psi = \frac{d\theta}{dx} = \frac{R^2}{h}\frac{d^2 w}{dr^2};\ S_r = \frac{\widetilde{N}_r R^3}{Eh^3};\ S_\theta = \frac{\widetilde{N}_\theta R^3}{Eh^3} \qquad (22)$$

where S_r, S_θ are the in-plane stress resultants induced by the lateral load

$$S_r = N_r - N_0; \qquad S_\theta = N_\theta - N_0, \qquad (23)$$

The non-dimensional forms of (19 - 21) are

$$\frac{d^2\theta}{dx^2} + \frac{1}{x}\frac{d\theta}{dx} - \left(k^2 + \frac{1}{x^2}\right)\theta - 12\left(1-\nu^2\right)S_r\theta = 6\left(1-\nu^2\right)Px ; \qquad (24)$$

$$\frac{dS_r}{dx} + \frac{S_r - S_\theta}{x} = 0 ; \qquad (25) \qquad\qquad \frac{dS_\theta}{dx} + \frac{S_r - S_\theta}{x} + \frac{1}{2x}\theta = 0 \qquad (26)$$

where a non-dimensional tension parameter k, and a loading parameter P, have been introduced as,

$$k = \sqrt{\frac{N_0 R^2}{D}} = \frac{R}{h}\sqrt{\frac{12\left(1-\nu^2\right)N_0}{Eh}} ; \qquad\qquad P = \frac{p_0 R^4}{Eh^4} . \qquad (27)$$

The most applications of circular plates are considered to the clamped conditions. An additional boundary conditions is required that the amount of stretch, u, at the edge of a circular plate is zero. This is further cast in terms of circumferential strain through the center thickness of the plate and the built-in residual strain. The following boundary conditions for the circular plate are

$$\xi = 0 \qquad \theta = \frac{dW}{d\xi} = 0 \qquad S_r = S_\theta \qquad (28,a)$$

$$\xi = 1 \qquad \theta = \frac{dW}{d\xi} = 0 \qquad U = \frac{h}{R}\left(S_\theta - \nu S_r\right)\xi\Big|_{\xi=1} = 0 . \qquad (28,b)$$

2.1. Linear theory;

In our case for small defections, the mid-plane load S_r is assumed small, so that the nonlinear $S_r\theta$ term in (24) can be neglected. Multiplying (24) by x^2 then leads to the linear equation ,

$$x^2\frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - \left(x^2 k^2 + 1\right)\theta = 6\left(1-\nu^2\right)Px^3 \qquad (29)$$

which is a modified Bessel equation possessing the general solution,

$$\theta(x) = C_1 I_1(kx) + C_2 K_1(kx) - 6\left(1-\nu^2\right)\frac{Px}{k^2} \qquad (30)$$

where I_1(kx) and K_1(kx) are modified Bessel functions of the first kind and second kind, respectively.

The requirement of a bounded solution at $x = 0$ and implementation the boundary conditions (27- 28) yields,

$$W(x) = 3\left(1-\nu^2\right)P\left[\frac{2\left[I_0(k\,x) - I_0(k)\right]}{k^3 I_1(k)} + \frac{1-x^2}{k^2}\right]. \quad (31)$$

Two limiting cases are of interest here. For the case of a pure plate (k = 0), (29) with $\theta = dW/d\xi$ becomes an ordinary differential equation possessing the following solution:

$$W(x) = \frac{1}{16}\left(1-\nu^2\right)P\left(1-x^2\right)^2. \quad (32)$$

For small deflection, when $N_0 = 0$, work [9] shows that deflection, W, of a clamped circular plate under a uniform applied pressure p_0 is given by equation (32).

For the other case of a pure membrane, $k \to \infty$, only the $k^2x^2\theta$ term remains on the left hand side of Equation 29. The resulting equation is integrated to yield,

$$\theta(x) = -6\left(1-\nu^2\right)\frac{Px^2}{k^2} \quad \text{and} \quad W(x) = 3\left(1-\nu^2\right)\frac{P}{k^2}\left(1-x^2\right). \quad (33)$$

2.2. Nonlinear theory

For large deflections (W >> 1), nonlinear behavior arises when the incremental mid-plane forces, $\widetilde{N}_r$, $\widetilde{N}_\theta$, are no longer negligible. This behavior is manifested by the θ^2 term in (25) changing the mid-plane force S_r, which in turn affects the defection equations through the nonlinear $S_r\theta$ term in (24).

A convenient way of numerically solving this nonlinear system is to combine (24 - 25) and recast (23 - 25) as two coupled, second-order equations in the variables θ and S_r,

$$x^2\frac{d^2\theta}{dx^2} + x\frac{d\theta}{dx} - \left[1 + x^2\left(k^2 + 11{,}12S_r\right)\right]\theta = 6\left(1-\nu^2\right)Px^2; \quad (34) \qquad x^2\frac{d^2S_r}{dx^2} + 3x\frac{dS_r}{dx} = -\frac{1}{2}\theta^2. \quad (35)$$

Knowing the expressions of transverse and radial displacements, w and u, are set expressions for radial and tangential stresses. Equations (34) and (35) were then discretized using 2nd-order accurate central- difference schemes, while the derivative and mixed boundary conditions were implemented by using 2nd-order accurate forward- and backward-dfferencing schemes, respectively. In the table 2 are presents the numerical results

2.3 Finite element method

Finite element analysis is a valuable design tool for the circular plate, since it generally gives more accurate results than analytical solutions. A circular plate with fixed edges and a constant residual stress was modeled in ANSYS.

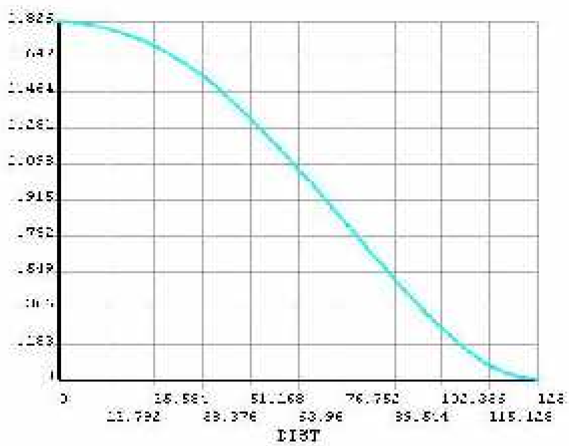

Figure 2. Representation at the tenth iteration of the deflection

The finite element analysis was used finite element type Shell 63. Numerical results are presented for a circular plate of radius, $R = 125\ mm$, thickness, $h = 3mm$, Poisson ratio, $\nu = 0.3$, Young modulus, $E = 2.1\cdot10^5 MPa$, and loading, $p_0 = 0.3\ MPa$. Running the data was done in 10 steps. The results are summarized in the table 2 and in figure 2 it is representation at the tenth iteration of the deflection.

3. THE EXPERIMENTAL DEVICE

Figure 3 presents the loading device for a thin plane plate. Between the middle segment (1) and the rings (2) and (3) two identical thin plates models were fixed (7) and (8). The plate (7) is used as a model for the determination of the experimental models. The plate (8) was used to block the rotation of the rings (2) and (3) as well as the body (1) during solicitation, thus realizing a a clamping simulation as close to reality as possible. The rings (2) and (3) fix the two identical models from plates in the body (1) with the help of 36 screws. The body (1) presents on both sides the widening F and G which communicates one with the other. These spaces receive the oil subjected to hydraulic pressure from the hydraulic pump (17) with the help of a connection pipe (14).

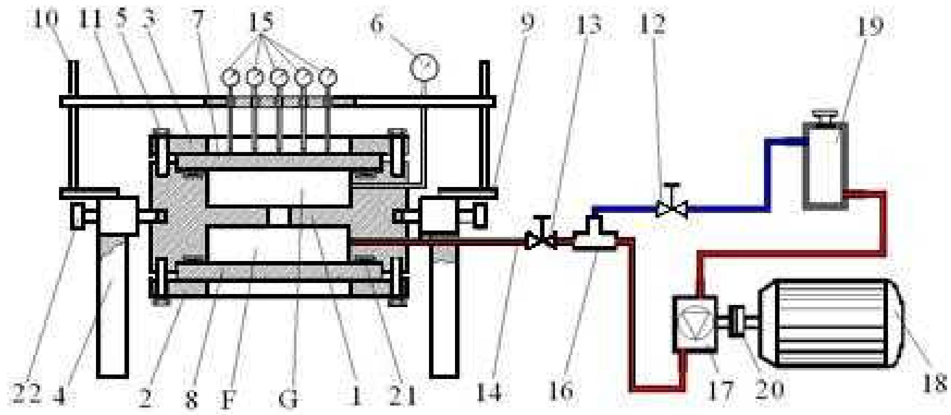

Figure 3 Plate fixing and solicitation device

The pressure for the oil in the spaces F and G can be read with the help of the manometer (6). The transversal displacement of the plate has been determined by the micrometer with dial indicator (15) with a precision of 0.01 mm positioned in the radial direction with the help of a device (9) (fig.1). The body of the device (1) is fixed on the support (4). The placement of the microcomparator on the plate in radial direction in the table 1 is established.

In the table 2 are presents the numerical, experimental and finite element results and they are compared between them.

Table 1 The placement of the microcomparator [figure]

Microcomparator	1	2	3	4	5
Radius	0	54 mm	72 mm	90 mm	108 mm

Table 2 Determined numerical results, analytical experimental and with MEF

Placement of micrometer	Theory – numerical	MEF	Experimental 0.05 [Mpa]	MEF-EXP	Theory – experimental	Teoretical – MEF	Theory – numerical	MEF	Experimental 0.075 [Mpa]	MEF-Experimental	Theory – experimental	Theory-MEF
1	0,449	0,436	0,44	1,009%	0,980%	0,971%	0,787	0,781	0,78	0,999%	0,991%	0,992%
	0,406	0,387	-	-	-	0,953%	0,722	0,693	-	-	-	0,960%
2	0,354	0,339	0,32	0,944%	0,904%	0,958%	0,636	0,606	0,59	0,974%	0,928%	0,953%
	0,298	0,291	-	-	-	0,977%	0,528	0,521	-	-	-	0,987%
3	0,241	0,242	0,22	0,909%	0,913%	1,004%	0,421	0,433	0,42	0,970%	0,998%	1,029%
	0,184	0,193	-	-	-	1,049%	0,324	0,346	-	-	-	1,068%
4	0,131	0,145	0,14	0,966%	1,069%	1,107%	0,228	0,261	0,25	0,958%	1,096%	1,145%
	0,079	0,096	-	-	-	1,215%	0,141	0,173	-	-	-	1,227%
5	0,039	0,048	0,04	0,833%	1,026%	1,231%	0,068	0,086	0,08	0,930%	1,176%	1,265%
	0,011	0,023	-	-	-	2,091%	0,019	0,037	-	-	-	1,947%
6	0	0	0	0,000%	0,000%	0,000%	0	0	0	0,000%	0,000%	0,000%

4. CONCLUSION

This paper presents an approximate analytical method for the analysis of the bending of the thin circular plates with large displacements. The method is developed for a simply clamped plate subjected to a uniformly distributed load, as presented in this paper as example. The analytical method presented can be utilized for the analysis of the bending of circular plates with various rim conditions. The results obtained with the analytical method are then compared with the experimental results and the finite element method provides satisfactory results. The transversal displacements of the plate have been measured with the help of micro comparators by using the device in figure 3.

In this paper is developed a method for analyzing experimental bending thin plates with large displacements.. It is considered circular plate embedded in the shape required to uniformly distributed pressure. Se folosesc microcomparatoare pentru măsurarea deplasărilor transversale şi mărci tensometrice pentru măsurarea deformaţiilor. Microcomparatoare used to measure transverse displacements and strain gauges for measuring deformation marks.

To verify experimental results were used numerical method namely finite element method (ANSYS).

3. Results obtained on models created using finite element programs and experimental results for the few cases studied are satisfactory.

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