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A FEW REMARKS ABOUT THE PRESSURE CORRECTION METHOD

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Abstract: *In this note, we will present some considerations about a solution of Couette flow using the pressure correction technique. This gives us the opportunity to examine the behavior of the pressure correction method to an incompressible flow problem.*

1. INTRODUCTION

The incompressible Couette flow represents an exact analytical solution of the Navier-Stokes equations. Couette flow is perhaps the simplest of all viscous flow, while at the same time retaining much of the same psyhical characteristics of a more complicated boundary layer-flow.
During this paper, we'll deal with the two-dimensional Navier-Stokes equations for incompressible flow and set up a solution of these for the incompressible flow between two parallel plates in relative motion to each other using the pressure correction method. This method is an iterative approach and we'll set un the initial condition to be a two-dimensional flow field.

2. THE PSYHICAL PROBLEM A BRIEFLY PRESENTATION

Couette flow is defined as follows.
Let us consider the viscous flow between two parallel plates separated by the vertical distance D.
The upper plate is moving at the velocity u_e and the lower plate is stationary (i.e. the velocity is $u \equiv 0$).
The flow in the xy plane is sketched in Fig.1

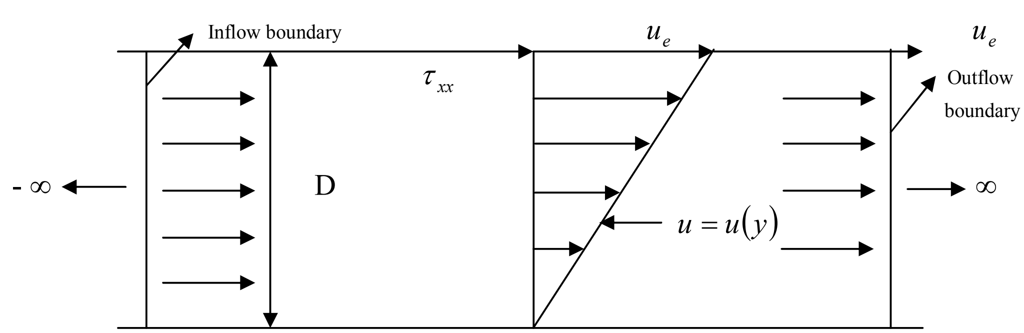

Figure 1

The flow field between the two plates is driven exclusively by the shear stress exerted on the fluid by the moving upper plate, resulting in a velocity profile across the flow, $u = u(y)$, as sketched in Fig.1.

Since there is no beginning or end of this flow, the flow-field variables must be independent of x (i.e. $\frac{\partial \cdot}{\partial x} \equiv 0$ for all quantities).

Another physical characteristic is that there's no vertical component of velocity anywhere; this states that the stream lines for Couette flow are straight, parallel stream-lines.

For Couette flow, there are no pressure gradients in either the x or y direction.

With these previous remarks and using the x- (resp. $y-$) momentum equation, we obtain the governing equation for incompressible, constant-temperature Couette flow:

$$\frac{\partial^2 u}{\partial y^2} = 0 \qquad \textbf{(1)}$$

Integrating twice with respect to y, we have the exact analytical solution:

$$u = c_1 y + c_2 \qquad \textbf{(2)}$$

(where c_1, c_2 are constants of integration).

We find their values by appling the boundary conditions:

$$u(0) = 0 \Rightarrow c_2 = 0$$

$$u(D) = u_e \Rightarrow c_1 = \frac{u_e}{D} \qquad \textbf{(3)}$$

Thes,

$$\frac{u}{u_e} = \frac{y}{D} \qquad \textbf{(4)}$$

represents the exact analytical solution for the velocity profile for incompressible Couette flow (which is a linear profile).

3. SOME COMMENTS ABOUT THE PHILOSOPHY OF THE PRESSURE CORRECTION METHOD

The pressure correction technique has been developed for practical engineering solutions by Patankar and Spalding and it is embodied in an algorithm called S I M P L E (semi-implicit method for pressure-linked equation), which has found widespread application for both compressible and incompressible flows.

This technique is basically an iterative approach, where some innovative physical reasoning is used to construct the next iteration from the results of the previous iteration.

The thought process is as follows:

1. Start the iterative process by guessing the pressure field; denote the guessed pressure by p^*.
2. From the momentum equations, using the values of p^*, solve for u, v, w. Since these velocities are those associated with the values of p^*, denote them by u^*, v^*, w^*.
3. Using the continuity equation, construct a pressure correction p' i.e.

   $$p = p^* + p' \qquad \textbf{(5)}$$

   (so we obtain the velocity field more into agreement with the continuity equation). Corresponding velocities corrections (obtained from p') are: u', v', w' such that:

   $$u = u^* + u', \ \mathrm{v} = \mathrm{v}^* + v', \ \mathrm{w} = \mathrm{w}^* + w' \qquad \textbf{(6)}$$

4.In Eq.5, designate the new value p on the left side as the new value of p^*.

Return to step 2 and repeat the process until a velocity field is found does satisfy the continuity equation. When this is achieved, the correct flow field is at hand.

4. THE PRESSURE CORRECTION METHOD FOR THE COUETTE FLOW

Now we apply the pressure correction method to the solution of the incompressible, viscous flow between two parallel plates as in Fig.1. Although the plates are theoretically infinite in extent, the computational domain is finite with length, L and height D.

Let us consider $L = 0{,}5$ and $D = 0{,}01$.

The fluid is air at standard sea level conditions with a density $g = 0{,}002377$.

We consider the case of a low velocity (we set $u_e = 1$) and there's no doubt that the flow is incompressible, in this case. The Reynolds number for this case is $63{,}6$.

Based on the reasons from the general case, we choose a staggered grid (with 3 systems of grid points: the solid points are where p is calculated, the open points, where u is calculated and the points denoted by x, where v is calculated), as in the following figure:

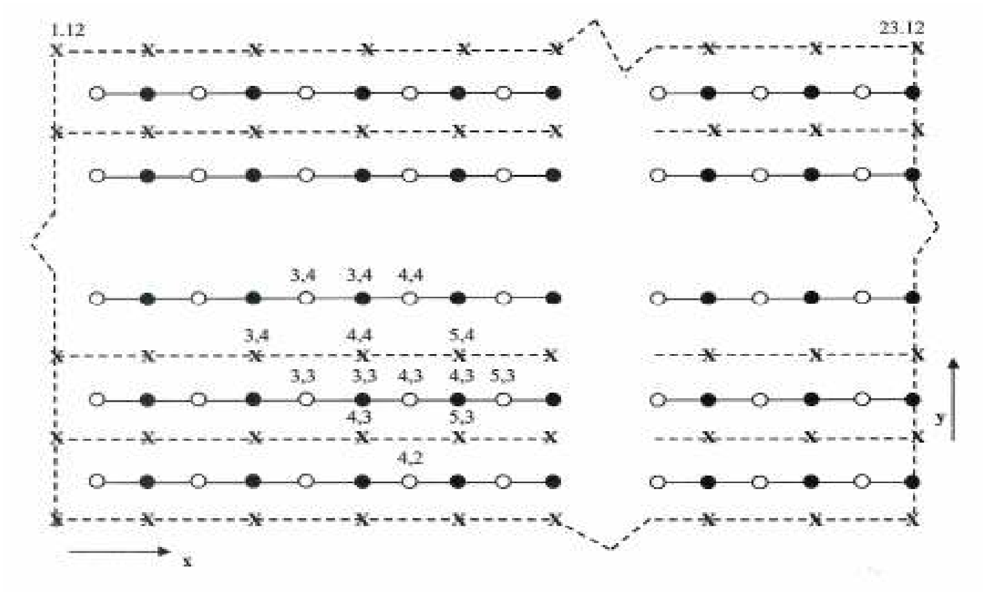

Figure 2

Note that the "p points" run from 1 to 21 in the x direction, from 1 to 11 in the y direction; the "u points" from 1 to 22 in the x direction, from 1 to 11 in the y direction; the "v points" from 1 to 23 in the x direction and from 1 to 12 in the y direction.

Now, we need, as this method is an iterative method, initial conditions for the flow variables in order to start the iterative process. The choice is arbitrary.

For the present case, we set:

$$u = v = 0,\ p^* = p' = 0 \text{ for all interior points} \quad \textbf{(7)}$$

except for point $(i, j) = (15{,}5)$, where $v = 0{,}5$.

The boundary values are:

$$\left.\begin{array}{ll} u = u_e,\ v = 0 & \text{- at the upper wall} \\ u = v = 0 & \text{- at the lower wall} \\ p' = 0,\ v = 0 & \text{- at the in flow boundary} \\ p' = 0 & \text{- at the out flow boundary} \end{array}\right. \quad \textbf{(8)}$$

Step 1: Guess at value of p^* at all interior grid points. Also, arbitrarily set values of $(\rho u^*)^n$ and $(\rho v^*)^n$ at all the appropriate grid points.

$p^*, \rho u^*, \rho v^* \equiv 0$ for the beginning of the iterative process; $u_e = 1$ at the upper wall; $v^*_{15,5} = 0{,}5$ at the velocity spike (to produce a two-dimensional flow during the iterative process).

Step 2: Determine $\left(\rho u^*\right)^{n+1}$ and $\left(\rho v^*\right)^{n+1}$.
We use the pressure grid point $(3,3)$, as a focus.
From the general case, we obtain:

I. $$\left(\rho u^*\right)_{4,3}^{n+1} = \left(\rho u^*\right)_{4,3}^{n} + A^*\Delta t - \frac{\Delta t}{\Delta x}\left(p_{4,3}^* - p_{3,3}^*\right) \quad \textbf{(9)}$$

where:

$$A^* = -\left[\frac{\left(\rho u^2\right)_{5,3}^n - \left(\rho u^2\right)_{3,3}^n}{2\Delta x} + \frac{\left(\rho u\bar{v}\right)_{4,4}^n - \left(\rho u\bar{\bar{v}}\right)_{4,2}}{2\Delta y}\right] +$$
$$+\left[\frac{u_{5,3}^n - 2u_{4,3}^n + u_{3,3}^n}{(\Delta x)^2} + \frac{u_{4,4}^n - 2u_{4,3}^n + u_{4,2}^n}{(\Delta y)^2}\right] \quad \textbf{(10)}$$

$$\bar{v} = \frac{1}{2}\left(v_{4,4}^n + v_{5,4}^n\right),\ \ \bar{\bar{v}} = \frac{1}{2}\left(v_{4,3}^n + v_{5,3}^n\right) \quad \textbf{(11)}$$

II. $$\left(\rho v^*\right)_{4,4}^{n+1} = \left(\rho v^*\right)_{4,4}^{n} + B^*\Delta t - \frac{\Delta t}{\Delta y}\left(p_{3,4}^* - p_{3,3}^*\right) \quad \textbf{(12)}$$

where:

$$B^* = -\left[\frac{\left(\rho v\bar{u}\right)_{5,4}^n - \left(\rho v\bar{\bar{u}}\right)_{3,4}^n}{2\Delta x} + \frac{\left(\rho v^2\right)_{4,5}^n - \left(\rho v^2\right)_{4,3}}{2\Delta y}\right] +$$
$$+\left[\frac{v_{5,4}^n - 2v_{4,4}^n + v_{3,4}^n}{(\Delta x)^2} + \frac{v_{4,5}^n - 2v_{4,4}^n + v_{4,3}^n}{(\Delta y)^2}\right] \quad \textbf{(13)}$$

$$\bar{u} = \frac{1}{2}\left(u_{4,3}^n + u_{4,4}^n\right),\ \ \bar{\bar{u}} = \frac{1}{2}\left(u_{3,3}^n + u_{3,4}^n\right) \quad \textbf{(14)}$$

After $\left(\rho u^*\right)$ and $\left(\rho v^*\right)$ are obtained for all the interior grid points, by dividing these values by ρ, we obtain u^* and v^*.
At the inflow boundary,

$$u_{1,j}^* = u_{2,j}^*,\ (\forall)j \quad \textbf{(15)}$$

$$u_{22,j}^* = u_{21,j}^*,\ v_{23,j}^* = v_{22,j}^*(\forall)j \quad \textbf{(16)}$$

In the above equations,

$$\Delta x = \frac{0{,}5}{20} = 0{,}025, \quad \Delta y = \frac{0{,}01}{10} = 0{,}001, \quad \Delta t = 0{,}001 \quad \textbf{(17)}$$

(The value of Δt was chosen somewhat arbitrarily; Δt plays the role of the "relaxation factor"; the value chosen was consider to be acceptable for this calculation)
Step 3: Solve for p' from the pressure correction formula focusing on the pressure grid point (3,3):

$$p'_{3,3} = -\frac{1}{a}\left(bp'_{4,3} + bp'_{2,3} + cp'_{3,4} + cp'_{3,2} + d\right) \quad \textbf{(18)}$$

where:

$$a = 2\Delta t\left[\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}\right];\ \ \mathrm{b} = -\frac{\Delta \mathrm{t}}{(\Delta \mathrm{x})^2};\ \ \mathrm{c} = -\frac{\Delta t}{(\Delta y)^2};$$
$$d = \frac{1}{\Delta x}\left[\left(\rho u^*\right)_{4,3} - \left(\rho u^*\right)_{3,3}\right] + \frac{1}{\Delta y}\left[\left(\rho v^*\right)_{4,4} - \left(\rho v^*\right)_{4,3}\right] \quad \textbf{(19)}$$

Note that eq. like (18) are solved for $p'_{i,j}$ at every interior grid point using a relaxation technique. (In this case, there are over 200 relaxation steps necessarily that the values of $p_{i,j}$ have converged).

Step 4: calculate p^{n+1} at all internal grid points, from:

$$p_{i,j}^{n+1} = \left(p_{i,j}^{*}\right)^{n} + \alpha_p p' \qquad \textbf{(20)}$$

where α_p constitutes an underrelaxation factor. We chose $\alpha_p = 0{,}1$.

Step 5: Designate the value of $p_{i,j}^{n+1}$ (obtained from the Step 4) as the new values of $\left(p^{*}\right)^{n}$.
Return to Step 2 and repeat Step 2 to Step 5 until convergence is achieved.
(It takes over 300 iterations in the present case).

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