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NEO–HOOKEAN SUSPENSION FOR A HALF OF AN AUTOMOBILE

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Abstract : *In this paper we shall present a four degrees of freedom model for a half of an automobile. The suspensions consist of linear and non-linear elements, the non-linear being of neo-Hookean type. For this model we obtain the equations of motion, the equilibrium positions and we study the stability of the equilibrium positions. Finally, a numerical example is also presented.*
Keywords (TNR 9 pt Bold): *neo-Hookean, suspension, automobile, stability*

1. MATHEMATICAL MODEL

We shall now present the study of the motion for four degrees of freedom system that models a half of an automobile. The model is presented in figure 1. This model consists of the masses m_1 and m_2, which mark the wheels of the automobile, masses linked to the ground by linear elastic springs of stiffness k_1 and k_2, respectively. By wheels is attached the chassis marked by the bar AB of mass M. The linking of the chassis is made by the non-linear neo–Hookean elastic elements by elastic stiffness d_1, e_1, respectively d_2, e_2. The elastic force that appears in such element is given by

$$F = d_i z_i - \frac{e_i}{z_i^2}, \qquad \textbf{(1)}$$

where $i = \overline{1,2}$, z_i marks the elongation of the respective element, and $d_i > 0$, $e_1 > 0$, $i = \overline{1,2}$.

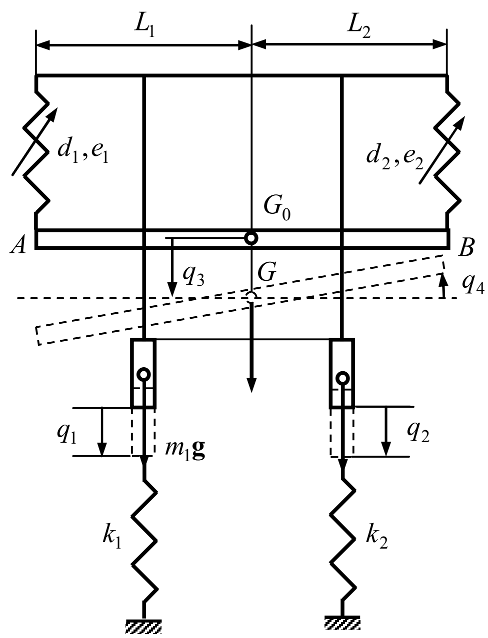

Fig. 10: The mathematical model

The four degrees of freedom of the system were selected as follows: q_1, q_2 the elongations of the linear springs, q_3 the displacement in the vertical direction of the gravity centre G of the chassis and q_4 the rotation of the chassis with respect to the horizontal.
We assume that there are known the dimensions L_1 and L_2 that define the position of the gravity centre G of the chassis with respect to the two wheels and J the moment of the inertia with respect to a horizontal axis that passes through its gravity centre.

2. THE EQUATIONS OF MOTION

The kinetic energy of the system has the expression

$$T = \frac{1}{2} m_1 \dot{q}_1^2 + \frac{1}{2} m_2 \dot{q}_2^2 + \frac{1}{2} M \dot{q}_3^2 + \frac{1}{2} J \dot{q}_4^2 . \tag{2}$$

The forces, which appear in the system, derive from a potential, hence the potential energy reads

$$V = \frac{1}{2} k_1 q_1^2 - m_1 g q_1 + \frac{1}{2} k_2 q_2^2 - m_2 g q_2 + \frac{1}{2} d_1 \left(L_1 q_4 - q_1 + q_3\right)^2 + \frac{e_1}{L_1 q_4 - q_1 + q_3} + \frac{1}{2} d_2 \left(q_3 - L_2 q_4 - q_2\right)^2 + \frac{e_2}{q_3 - L_2 q_4 - q_2} - M g q_3 , \tag{3}$$

g being the gravitational acceleration.
We successively calculate

$$\frac{\partial T}{\partial \dot{q}_1} = m_1 \dot{q}_1 ; \ \frac{\partial T}{\partial \dot{q}_2} = m_2 \dot{q}_2 ; \ \frac{\partial T}{\partial \dot{q}_3} = M \dot{q}_3 ; \ \frac{\partial T}{\partial \dot{q}_4} = J \dot{q}_4 , \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_1}\right) = m_1 \ddot{q}_1 ; \ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_2}\right) = m_2 \ddot{q}_2 ; \ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_3}\right) = M \ddot{q}_3 ; \ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_4}\right) = J \ddot{q}_4 , \tag{5}$$

$$\frac{\partial T}{\partial q_1} = 0 ; \ \frac{\partial T}{\partial q_2} = 0 ; \ \frac{\partial T}{\partial q_3} = 0 ; \ \frac{\partial T}{\partial q_4} = 0 , \tag{6}$$

$$\frac{\partial V}{\partial q_1} = k_1 q_1 - m_1 g - d_1 \left(L_1 q_4 - q_1 + q_3\right) + \frac{e_1}{\left(L_1 q_4 - q_1 + q_3\right)^2} , \tag{7a}$$

$$\frac{\partial V}{\partial q_2} = k_2 q_2 - m_2 g - d_2 \left(q_3 - L_2 q_4 - q_2\right) + \frac{e_2}{\left(q_3 - L_2 q_4 - q_2\right)^2} , \tag{7b}$$

$$\frac{\partial V}{\partial q_3} = d_1 \left(L_1 q_4 - q_1 + q_3\right) - \frac{e_1}{\left(L_1 q_4 - q_1 + q_3\right)^2} + d_2 \left(q_3 - L_2 q_4 - q_2\right) - \frac{e_2}{\left(q_3 - L_2 q_4 - q_2\right)^2} - M g , \tag{7c}$$

$$\frac{\partial V}{\partial q_4} = L_1 d_1 \left(L_1 q_4 - q_1 + q_3\right) - \frac{L_1 e_1}{\left(L_1 q_4 - q_1 + q_3\right)^2} - L_2 d_2 \left(q_3 - L_2 q_4 - q_2\right) + \frac{L_2 e_2}{\left(q_3 - L_2 q_4 - q_2\right)^2} , \tag{7d}$$

such that the Lagrange equations read

$$m_1 \ddot{q}_1 + k_1 q_1 - m_1 g - d_1 \left(L_1 q_4 - q_1 + q_3\right) + \frac{e_1}{\left(L_1 q_4 - q_1 + q_3\right)^2} = 0 , \tag{6a}$$

$$m_2 \ddot{q}_2 + k_2 q_2 - m_2 g - d_2 \left(q_3 - L_2 q_4 - q_2\right) + \frac{e_2}{\left(q_3 - L_2 q_4 - q_2\right)^2} = 0 , \tag{8b}$$

$$M \ddot{q}_3 + d_1 \left(L_1 q_4 - q_1 + q_3\right) - \frac{e_1}{\left(L_1 q_4 - q_1 + q_3\right)^2} + d_2 \left(q_3 - L_2 q_4 - q_2\right) - \frac{e_2}{\left(q_3 - L_2 q_4 - q_2\right)^2} - M g = 0 , \tag{8c}$$

$$J \ddot{q}_4 + L_1 d_1 \left(L_1 q_4 - q_1 + q_3\right) - \frac{L_1 e_1}{\left(L_1 q_4 - q_1 + q_3\right)^2} - L_2 d_2 \left(q_3 - L_2 q_4 - q_2\right) + \frac{L_2 e_2}{\left(q_3 - L_2 q_4 - q_2\right)^2} = 0 . \tag{8d}$$

Let us denote

$$\xi_1 = q_1 ; \ \xi_2 = q_2 ; \ \xi_3 = q_3 ; \ \xi_4 = q_4 ; \ \xi_5 = \dot{q}_1 ; \ \xi_6 = \dot{q}_2 ; \ \xi_7 = \dot{q}_3 , \ \xi_8 = \dot{q}_4 . \tag{9}$$

obtaining a system of eight first order non-linear differential equations

$$\frac{\mathrm{d}\xi_1}{\mathrm{d}t} = \xi_5 ; \ \frac{\mathrm{d}\xi_2}{\mathrm{d}t} = \xi_6 ; \ \frac{\mathrm{d}\xi_3}{\mathrm{d}t} = \xi_7 ; \ \frac{\mathrm{d}\xi_4}{\mathrm{d}t} = \xi_8 , \tag{10a}$$

$$\frac{\mathrm{d}\xi_5}{\mathrm{d}t} = \frac{1}{m_1}\left[- k_1 \xi_1 + m_1 g + d_1 \left(L_1 \xi_4 - \xi_1 + \xi_3\right) - \frac{e_1}{\left(L_1 \xi_4 - \xi_1 + \xi_3\right)^2} \right] . \tag{10b}$$

$$\frac{\mathrm{d}\xi_6}{\mathrm{d}t} = \frac{1}{m_2}\left[-k_2\xi_2 + m_2 g + d_2(\xi_3 - L_2\xi_4 - \xi_2) - \frac{e_2}{(\xi_3 - L_2\xi_4 - \xi_2)^2}\right], \tag{10c}$$

$$\begin{aligned}\frac{\mathrm{d}\xi_7}{\mathrm{d}t} = \frac{1}{M}\Bigg[&-d_1(L_1\xi_4 - \xi_1 + \xi_3) + \frac{e_1}{(L_1\xi_4 - \xi_1 + \xi_3)^2}\\ &- d_2(\xi_3 - L_2\xi_4 - \xi_2) + \frac{e_2}{(\xi_3 - L_2\xi_4 - \xi_2)^2} + Mg\Bigg],\end{aligned} \tag{10d}$$

$$\begin{aligned}\frac{\mathrm{d}\xi_8}{\mathrm{d}t} = \frac{1}{J}\Bigg[&-L_1 d_1(L_1\xi_4 - \xi_1 + \xi_3) + \frac{L_1 e_1}{(L_1\xi_4 - \xi_1 + \xi_3)^2}\\ &+ L_2 d_2(\xi_3 - L_2\xi_4 - \xi_2) - \frac{L_2 e_2}{(\xi_3 - L_2\xi_4 - \xi_2)^2}\Bigg].\end{aligned} \tag{10e}$$

3. THE EQUILIBRIUM POSITIONS

These are obtained at the intersection of the nullclines, resulting the system

$$\xi_5 = 0\,;\ \xi_6 = 0\,;\ \xi_7 = 0\,;\ \xi_8 = 0, \tag{11}$$

$$-k_1\xi_1 + m_1 g + d_1(L_1\xi_4 - \xi_1 + \xi_3) - \frac{e_1}{(L_1\xi_4 - \xi_1 + \xi_3)^2} = 0, \tag{12a}$$

$$-k_2\xi_2 + m_2 g + d_2(\xi_3 - L_2\xi_4 - \xi_2) - \frac{e_2}{(\xi_3 - L_2\xi_4 - \xi_2)^2} = 0, \tag{12b}$$

$$-d_1(L_1\xi_4 - \xi_1 + \xi_3) + \frac{e_1}{(L_1\xi_4 - \xi_1 + \xi_3)^2} - d_2(\xi_3 - L_2\xi_4 - \xi_2) + \frac{e_2}{(\xi_3 - L_2\xi_4 - \xi_2)^2} + Mg = 0, \tag{12c}$$

$$-L_1 d_1(L_1\xi_4 - \xi_1 + \xi_3) + \frac{L_1 e_1}{(L_1\xi_4 - \xi_1 + \xi_3)^2} + L_2 d_2(\xi_3 - L_2\xi_4 - \xi_2) - \frac{L_2 e_2}{(\xi_3 - L_2\xi_4 - \xi_2)^2} \tag{12d}$$

Adding the first three relations (12), one obtains the equation

$$-k_1\xi_1 - k_2\xi_2 + (m_1 + m_2 + M)g = 0. \tag{13}$$

Multiplying the first relation (12) by L_1, the second relation (12) by $-L_2$ and summing the results at the last expression (12), we deduce

$$-L_1 k_1\xi_1 + L_2 k_2\xi_2 + (L_1 m_1 - L_2 m_2)g = 0. \tag{14}$$

The relations (13) and (14) form a linear system of two equations with two unknowns (ξ_1 and ξ_2)

$$k_1\xi_1 + k_2\xi_2 = (m_1 + m_2 + M)g\,;\ L_1 k_1\xi_1 - L_2 k_2\xi_2 = (L_1 m_1 - L_2 m_2)g, \tag{15}$$

the solution of this system being

$$\xi_1 = \frac{\begin{vmatrix}(m_1 + m_2 + M)g & k_2\\ (L_1 m_1 - L_2 m_2)g & -L_2 k_2\end{vmatrix}}{\begin{vmatrix}k_1 & k_2\\ L_1 k_1 & -L_2 k_2\end{vmatrix}} = \frac{m_1(L_1 + L_2) + L_2 M}{(L_1 + L_2)k_1}\,g, \tag{16a}$$

$$\xi_2 = \frac{\begin{vmatrix}k_1 & (m_1 + m_2 + M)g\\ L_1 k_1 & (L_1 m_1 - L_2 m_2)g\end{vmatrix}}{\begin{vmatrix}k_1 & k_2\\ L_1 k_1 & -L_2 k_2\end{vmatrix}} = \frac{m_2(L_1 + L_2) + L_1 M}{(L_1 + L_2)k_2}\,g \tag{16b}$$

We multiply now the third equation (12) by $-L_1$ and we add it to the last equation (12) obtaining

$$(L_1 + L_2)d_2(\xi_3 - L_2\xi_4 - \xi_2) - \frac{(L_1 + L_2)e_2}{(\xi_3 - L_2\xi_4 - \xi_2)^2} - L_1 Mg = 0 \tag{17}$$

or, equivalently,

$$(\xi_3 - L_2\xi_4 - \xi_2)^3 - \frac{L_1}{(L_1 + L_2)d_2}Mg(\xi_3 - L_2\xi_4 - \xi_2)^2 - \frac{e_2}{d_2} = 0. \tag{18}$$

We multiply the third equation (12) by L_2 and we add it to the last equation (12) resulting

$$-\left(L_1+L_2\right)d_1\left(\xi_3+L_1\xi_4-\xi_1\right)^3+\frac{\left(L_1+L_2\right)e_1}{\left(\xi_3+L_1\xi_4-\xi_1\right)^2}+L_2Mg=0 \quad \mathbf{(19)}$$

or, equivalently,

$$\left(\xi_3+L_1\xi_4-\xi_1\right)^3-\frac{L_2}{\left(L_1+L_2\right)d_1}Mg\left(\xi_3+L_1\xi_4-\xi_1\right)^2-\frac{e_1}{d_1}=0\,. \quad \mathbf{(20)}$$

Let us consider for the beginning the equation (18) and let us denote

$$z=\xi_3-L_2\xi_4-\xi_2\,;\ \alpha=\frac{L_1}{\left(L_1+L_2\right)d_1}Mg\,;\ \beta=\frac{e_2}{d_2}\,;\ \alpha>0\,;\ \beta>0\,, \quad \mathbf{(21)}$$

resulting the relation

$$z^3-\alpha z^2-\beta=0\,. \quad \mathbf{(22)}$$

In the sequence of the coefficients for the equation (22) there exists only one variation of sign and applying the Descartes theorem, it results that the equation (22) has only one positive real root. Making the change of variable $z\mapsto -z$, one obtains the equation

$$z^3+\alpha z^2+\beta=0 \quad \mathbf{(23)}$$

for which there exists no variation of sign in the sequence of the coefficients. Applying again the Descartes theorem, it results that the equation (23) has no positive real root and therefore the equation (22) has no negative real root. In the end, we obtained that the equation (22) has only one real root, thus the equation (18) has one real root, too. Let us denote this root by z_1. Proceeding in an analogous way, one deduces that the equation (20) has one real root and we denote this root by z_2. It results the system

$$\xi_3-L_2\xi_4-\xi_2=z_1\,;\ \xi_3+L_1\xi_4-\xi_1=z_2\,, \quad \mathbf{(24)}$$

for which the solution is

$$\xi_3=\frac{\begin{vmatrix} z_1+\xi_2 & -L_2 \\ z_2+\xi_1 & L_1\end{vmatrix}}{\begin{vmatrix} 1 & -L_2 \\ 1 & L_1\end{vmatrix}}=\frac{L_1\left(z_1+\xi_2\right)+L_2\left(z_2+\xi_1\right)}{L_1+L_2}\,, \quad \mathbf{(25a)}$$

respectively

$$\xi_4=\frac{\begin{vmatrix} 1 & z_1+\xi_2 \\ 1 & z_2+\xi_1\end{vmatrix}}{\begin{vmatrix} 1 & -L_2 \\ 1 & L_1\end{vmatrix}}=\frac{\left(z_2+\xi_1\right)-\left(z_1+\xi_2\right)}{L_1+L_2}\,. \quad \mathbf{(25b)}$$

We obtained that there exists only one equilibrium position defined by the relations (16) and (25).

4. STABILITY OF THE EQUILIBRIUM

Let us denote by f_i the expressions in the right-hand side of the relations (10) and let be

$$j_{kl}=\frac{\partial f_k}{\partial \xi_l}\,;\ k=\overline{1,8}\,;\ l=\overline{1,8}\,. \quad \mathbf{(26)}$$

We have

$$j_{11}=0\,;\ j_{12}=0\,;\ j_{13}=0\,;\ j_{14}=0\,;\ j_{15}=1\,;\ j_{16}=0\,;\ j_{17}=0\,;\ j_{18}=0\,, \quad \mathbf{(27)}$$

$$j_{21}=0\,;\ j_{22}=0\,;\ j_{23}=0\,;\ j_{24}=0\,;\ j_{25}=0\,;\ j_{26}=1\,;\ j_{27}=0\,;\ j_{28}=0\,, \quad \mathbf{(28)}$$

$$j_{31}=0\,;\ j_{32}=0\,;\ j_{33}=0\,;\ j_{34}=0\,;\ j_{35}=0\,;\ j_{36}=0\,;\ j_{37}=1\,;\ j_{38}=0\,, \quad \mathbf{(29)}$$

$$j_{41}=0\,;\ j_{42}=0\,;\ j_{43}=0\,;\ j_{44}=0\,;\ j_{45}=0\,;\ j_{46}=0\,;\ j_{47}=0\,;\ j_{48}=1\,, \quad \mathbf{(30)}$$

$$j_{51}=-\frac{k_1}{m_1}-\frac{d_1}{m_1}-\frac{2e_1}{m_1\left(L_1\xi_4-\xi_1+\xi_3\right)^3}\,;\qquad j_{52}=0\,;\qquad j_{53}=\frac{d_1}{m_1}+\frac{2e_1}{m_1\left(L_1\xi_4-\xi_1+\xi_3\right)^3}\,;$$

$$j_{54}=\frac{d_1L_1}{m_1}+\frac{2e_1L_1}{\left(L_1\xi_4-\xi_1+\xi_3\right)^3}\,;\ j_{55}=0\,;\ j_{56}=0\,;\ j_{57}=0\,;\ j_{58}=0\,, \quad \mathbf{(31)}$$

$$j_{61}=0\,;\qquad j_{62}=-\frac{k_2}{m_2}-\frac{d_2}{m_2}-\frac{2e_2}{m_2\left(\xi_3-L_2\xi_4-\xi_2\right)^3}\,;\qquad j_{63}=\frac{d_2}{m_2}+\frac{2e_2}{m_2\left(\xi_3-L_2\xi_4-\xi_2\right)^3}\,;$$

$$j_{64}=-\frac{d_2L_2}{m_2}-\frac{2e_2L_2}{m_2\left(\xi_3-L_2\xi_4-\xi_2\right)^3}\,;\ j_{65}=0\,;\ j_{66}=0\,;\ j_{67}=0\,;\ j_{68}=0\,, \quad \mathbf{(32)}$$

$$j_{71} = \frac{d_1}{M} + \frac{2e_1}{M(L_1\xi_4 - \xi_1 + \xi_3)^3}; \qquad j_{72} = \frac{d_2}{M} + \frac{2e_2}{M(\xi_3 - L_2\xi_4 - \xi_2)^3};$$

$$j_{73} = -\frac{d_1}{M} - \frac{2e_1}{M(L_1\xi_4 - \xi_1 + \xi_3)^3} - \frac{d_2}{M} - \frac{2e_2}{M(\xi_3 - L_2\xi_4 - \xi_2)^3};$$

$$j_{74} = -\frac{d_1L_1}{M} - \frac{2L_1e_1}{M(L_1\xi_4 - \xi_1 + \xi_3)^3} + \frac{d_2L_2}{M} + \frac{2e_2L_2}{M(\xi_3 - L_2\xi_4 - \xi_2)^3}; \quad j_{75} = 0; \quad j_{76} = 0; \quad j_{77} = 0;$$

$$j_{78} = 0, \tag{33}$$

$$j_{81} = \frac{L_1d_1}{J} + \frac{2L_1e_1}{J(L_1\xi_4 - \xi_1 + \xi_3)^3}; \qquad j_{82} = -\frac{L_2d_2}{J} - \frac{2L_2e_2}{J(\xi_3 - L_2\xi_4 - \xi_2)^3};$$

$$j_{83} = -\frac{L_1d_1}{J} - \frac{2L_1e_1}{J(L_1\xi_4 - \xi_1 + \xi_3)^3} + \frac{L_2d_2}{J} + \frac{2L_2e_2}{J(\xi_3 - L_2\xi_4 - \xi_2)^3};$$

$$j_{84} = -\frac{L_1^2d_1}{J} - \frac{2L_1^2e_1}{J(L_1\xi_4 - \xi_1 + \xi_3)^3} - \frac{L_2^2d_2}{J} - \frac{2L_2^2e_2}{J(\xi_3 - L_2\xi_4 - \xi_2)^3}; \quad j_{85} = 0; \quad j_{86} = 0; \quad j_{87} = 0;$$

$$j_{88} = 0. \tag{34}$$

The characteristic equation

$$\det(\mathbf{J} - \lambda\mathbf{I}) = 0, \tag{35}$$

where $\mathbf{J}$ is the Jacobi matrix

$$\mathbf{J} = [j_{kl}]_{k,l=\overline{1,8}}, \tag{36}$$

and $\mathbf{I}$ is the eight-order unity matrix, reads

$$\begin{vmatrix} -\lambda & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\lambda & 0 & 0 & 0 & 1 \\ j_{51} & 0 & j_{53} & j_{54} & -\lambda & 0 & 0 & 0 \\ 0 & j_{62} & j_{63} & j_{64} & 0 & -\lambda & 0 & 0 \\ j_{71} & j_{72} & j_{73} & j_{74} & 0 & 0 & -\lambda & 0 \\ j_{81} & j_{82} & j_{83} & j_{84} & 0 & 0 & 0 & -\lambda \end{vmatrix} = 0. \tag{37}$$

Multiplying the columns five, six, seven and eight by λ and summing the obtained results to the columns one, two, three and four, respectively, one deduces the equation

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ j_{51} - \lambda^2 & 0 & j_{53} & j_{54} & -\lambda & 0 & 0 & 0 \\ 0 & j_{62} - \lambda^2 & j_{63} & j_{64} & 0 & -\lambda & 0 & 0 \\ j_{71} & j_{72} & j_{73} - \lambda^2 & j_{74} & 0 & 0 & -\lambda & 0 \\ j_{81} & j_{82} & j_{83} & j_{84} - \lambda^2 & 0 & 0 & 0 & -\lambda \end{vmatrix} = 0. \tag{38}$$

Developing the determinant after the lines one, two, three and four, it results

$$\begin{vmatrix} j_{51} - \lambda^2 & 0 & j_{53} & j_{54} \\ 0 & j_{62} - \lambda^2 & j_{63} & j_{64} \\ j_{71} & j_{72} & j_{73} - \lambda^2 & j_{74} \\ j_{81} & j_{82} & j_{83} & j_{84} - \lambda^2 \end{vmatrix} = 0 \tag{39}$$

or, equivalently,

$$(j_{51} - \lambda^2)\begin{vmatrix} j_{62} - \lambda^2 & j_{63} & j_{64} \\ j_{72} & j_{73} - \lambda^2 & j_{74} \\ j_{82} & j_{83} & j_{84} - \lambda^2 \end{vmatrix} + j_{53}\begin{vmatrix} 0 & j_{62} - \lambda^2 & j_{64} \\ j_{71} & j_{72} & j_{74} \\ j_{81} & j_{82} & j_{84} - \lambda^2 \end{vmatrix} - j_{54}\begin{vmatrix} 0 & j_{62} - \lambda^2 & j_{63} \\ j_{71} & j_{72} & j_{73} - \lambda^2 \\ j_{81} & j_{82} & j_{83} \end{vmatrix} = 0. \tag{40}$$

The relation (39) is a bi-square equation of the fourth order in the unknown λ^2. It also offers the condition for the equilibrium position to be stable or unstable because it imposes a relation of connectivity in the space of the parameters k_1, k_2, d_1, d_2, e_1 and e_2.

5. APPLICATION

Let us consider the practical case for which

$$k_1 = k_2 = 4 \cdot 10^5 \left[\mathrm{N}/\mathrm{m}\right];\ d_1 = d_2 = 5 \cdot 10^4 \left[\mathrm{N}/\mathrm{m}\right];\ e_1 = e_2 = 5\left[\mathrm{Nm}^2\right];\ L_1 = L_2 = 2[\mathrm{m}];\ M = 900[\mathrm{kg}];$$
$$m_1 = m_2 = 25[\mathrm{kg}];\ g = 10\left[\mathrm{m}/\mathrm{s}^2\right]. \tag{41}$$

The relations (16) offer

$$\xi_1 = q_1^{ech} = \frac{25 \cdot (2+2) + 2 \cdot 900}{(2+2) \cdot 4 \cdot 10^5} = 0.0011875[\mathrm{m}];\ \xi_2 = q_2^{ech} = \frac{25 \cdot (2+2) + 2 \cdot 900}{(2+2) \cdot 4 \cdot 10^5} = 0.0011875[\mathrm{m}]. \tag{42}$$

The equation (18) becomes

$$z^3 - \frac{2}{(2+2) \cdot 5 \cdot 10^4} \cdot 900 \cdot 10 z^2 - \frac{5}{5 \cdot 10^4} = 0 \tag{43}$$

wherefrom

$$z^3 - 0.09z^2 - 0.0001 = 0 \tag{44}$$

with the solution

$$z = z_1^{ech} = 0.1[\mathrm{m}]. \tag{45}$$

In an analogous way we find

$$z_2^{ech} = 0.1[\mathrm{m}]. \tag{46}$$

The expressions (25) offer

$$\xi_3 = q_3^{ech} = \frac{2(0.1 + 0.0011875) + 2(0.1 + 0.0011875)}{2+2} = 0.1011875[\mathrm{m}], \tag{47}$$

$$\xi_4 = \frac{2(0.1 + 0.0011875) + 2(0.1 + 0.0011875)}{2+2} = 0.1011875[\mathrm{m}]. \tag{48}$$

The partial derivatives read

$$j_{51} = -18400;\ j_{52} = 0;\ j_{53} = 2400;\ j_{54} = 4800, \tag{49}$$

$$j_{61} = 0;\ j_{62} = -18400;\ j_{63} = 2400;\ j_{64} = -4800, \tag{50}$$

$$j_{71} = 66.667;\ j_{72} = 66.667;\ j_{73} = -133.333;\ j_{74} = 0, \tag{51}$$

$$j_{81} = 100;\ j_{82} = -100;\ j_{83} = 0;\ j_{84} = -400. \tag{52}$$

Results the characteristic equation

$$\begin{vmatrix} -18400 - \lambda^2 & 0 & 2400 & 4800 \\ 0 & -18400 - \lambda^2 & 2400 & -4800 \\ 66.667 & 66.667 & -133.333 - \lambda^2 & 0 \\ 100 & -100 & -\lambda^2 & -400 - \lambda^2 \end{vmatrix} = 0, \tag{53}$$

wherefrom

$$\lambda^8 + 37333.333\lambda^6 + 356959994.7\lambda^4 + 1.5872 \cdot 10^{11}\lambda^2 + 1.36533 \cdot 10^{13} = 0. \tag{54}$$

We denote

$$\lambda^2 = \eta \tag{55}$$

and one obtains the four-order equation

$$\eta^4 + 37333.333\eta^3 + 356959994.7\eta^2 + 1.5872 \cdot 10^{11}\eta + 1.36533 \cdot 10^{13} = 0. \tag{56}$$

The solving of this equation is made by the Lobacevski–Graeffe method for which for the equation

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0 \tag{57}$$

the passing from the step p to the step $p+1$ takes place with the formulas

$$a_0^{(p+1)} = \left[a_0^{(p)}\right]^2;\qquad a_1^{(p+1)} = -\left\{\left[a_1^{(p)}\right]^2 - 2a_0^{(p)}a_2^{(p)}\right\};\qquad a_2^{(p+1)} = \left[a_2^{(p)}\right]^2 - 2a_1^{(p)}a_3^{(p)} + 2a_0^{(p)}a_4^{(p)};$$
$$a_3^{(p+1)} = -\left\{\left[a_3^{(p)}\right]^2 - 2a_2^{(p)}a_4^{(p)}\right\};\ a_4^{(p+1)} = \left[a_4^{(p)}\right]^2. \tag{58}$$

We shall create the next table.

Table 1: The solving of equation (168) by the Lobacevski–Graeffe method

Step	a_0	a_1	a_2	a_3	a_4
0	1	37333.333	356959997.7	$1.5872 \cdot 10^{11}$	$1.36533 \cdot 10^{13}$
1	1	–679857763.5	$1.156 \cdot 10^{17}$	$-1.54 \cdot 10^{22}$	$1.864 \cdot 10^{26}$
2	1	$-2.310 \cdot 10^{17}$	$1.33 \cdot 10^{34}$	$-1.95 \cdot 10^{44}$	$3.47 \cdot 10^{52}$
3	1	$-2.668 \cdot 10^{34}$	$1.78 \cdot 10^{68}$	$-3.73 \cdot 10^{88}$	$1.21 \cdot 10^{105}$

Let be the function

$$h : \mathbb{R} \to \mathbb{R}\,;\; h(\eta) = \eta^4 + 37333.333\eta^3 + 356959994.7\eta^2 + 1.5872 \cdot 10^{11}\eta + 1.36533 \cdot 10^{13} \quad \textbf{(59)}$$

for which

$$h'(\eta) = 4\eta^3 + 112000\eta^2 + 713919989.4\eta + 1.5872 \cdot 10^{11}, \quad \textbf{(60)}$$

$$h''(\eta) = 12\eta^2 + 224000\eta + 713919989.4\,. \quad \textbf{(61)}$$

The equation $h''(\eta) = 0$ has the roots

$$\eta_{1,2} = \frac{-224000 \pm \sqrt{224000^2 - 4 \cdot 12 \cdot 713919989.4}}{24}, \quad \textbf{(62)}$$

wherefrom

$$\eta_1 = -4078.07\,;\; \eta_2 = -14588.6\,. \quad \textbf{(63)}$$

In addition,

$$h'(\eta_1) = -1.161 \cdot 10^{12} < 0\,;\; h'(\eta_2) = 1.161 \cdot 10^{12} > 0\,, \quad \textbf{(64)}$$

such that the equation $h'(\eta) = 0$ has three distinct real roots. We also have

$$h(-230) \approx -4.42 \cdot 10^{12} < 0\,; \quad h(-1) \approx 1.36 \cdot 10^{13} > 0\,; \quad h(-18000) \approx -3.8 \cdot 10^{12} < 0\,; \quad h(-5000) \approx 4.1 \cdot 10^{15} > 0 \quad \textbf{(65)}$$

and therefore the equation $h(\eta) = 0$ has four distinct negative real roots.

From the table 1 we get

$$\eta_1 = -\sqrt[8]{-\frac{a_1^{(3)}}{a_0^{(3)}}} = -\sqrt[8]{\frac{2.66 \cdot 10^{34}}{1}} \approx -20096\,, \quad \textbf{(66a)}$$

$$\eta_2 = -\sqrt[8]{-\frac{a_2^{(3)}}{a_1^{(3)}}} = -\sqrt[8]{\frac{1.78 \cdot 10^{68}}{2.66 \cdot 10^{34}}} \approx -16812\,, \quad \textbf{(66b)}$$

$$\eta_3 = -\sqrt[8]{-\frac{a_3^{(3)}}{a_2^{(3)}}} = -\sqrt[8]{\frac{3.73 \cdot 10^{88}}{1.78 \cdot 10^{68}}} \approx -346.8\,, \quad \textbf{(66c)}$$

$$\eta_4 = -\sqrt[8]{-\frac{a_4^{(3)}}{a_3^{(3)}}} = -\sqrt[8]{\frac{1.21 \cdot 10^{105}}{3.73 \cdot 10^{88}}} \approx -115.82\,. \quad \textbf{(66d)}$$

Result the roots of the characteristic equation

$$\lambda_1 \approx 141.76i\,;\; \lambda_2 \approx -141.76i\,;\; \lambda_3 \approx 130.05i\,;\; \lambda_4 \approx -130.05i\,;\; \lambda_5 \approx 18.62i\,;\; \lambda_6 \approx -18.62i\,;\; \lambda_7 \approx 10.76i\,;\; \lambda_8 \approx -10.76i \quad \textbf{(67)}$$

and all of them are pure imaginary, the equilibrium being simply stable.

6. CONCLUSIONS

In our paper we presented a model for a half of an automobile with neo-Hookean suspensions. We obtained the equations of motion and the equilibrium position. We proved that there exists only one equilibrium position. For this equilibrium position we discussed the conditions for its stability. In the end we presented a numerical application and we solved it completely, obtaining that the equilibrium position is simple stable.

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